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Title: Contact geometry and singular Lagrangian systems

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Abstract

Analytical mechanics, along with other areas of physics, admits a natural formulation within the framework of differential geometry. More specifically, many mechanical systems with dissipation can be described using Herglotz's variational principle, which leads to a modified form of the Euler–Lagrange equations. These equations admit a geometric formulation within the framework of contact geometry. In the case of a regular contact Lagrangian, the Legendre transformation gives rise to an equivalent contact Hamiltonian formulation.

This master's thesis is devoted to the study of singular contact Lagrangian systems. To this end, we begin by reviewing the geometric foundations of both ordinary and contact mechanics, as well as the theory of singular Lagrangian systems. The main original contributions of this thesis are the development of a new formulation of the contact dynamical equations, the definition of the evolution operator K for singular contact systems, as well as a novel treatment of precontact structures. To illustrate the theory, we provide examples of singular contact systems based on modified versions of the conformal particle and the Cawley Lagrangians.

Resum

La mecànica analítica, juntament amb altres àrees de la física, admet una formulació natural dins el marc de la geometria diferencial. Més específicament, molts sistemes mecànics amb dissipació poden ser descrits mitjançant el principi variacional de Herglotz, que condueix a una forma modificada de les equacions d'Euler–Lagrange. Aquestes equacions admeten una formulació geomètrica en el context de la geometria de contacte. En el cas d'una lagrangiana de contacte regular, la transformació de Legendre dona lloc a una formulació hamiltoniana de contacte equivalent.

Aquest treball de fi de màster està dedicat a l'estudi dels sistemes lagrangians singulars de contacte. Amb aquesta finalitat, comencem revisant els fonaments geomètrics tant de la mecànica ordinària com de la mecànica de contacte, així com de la teoria dels sistemes lagrangians singulars. Les principals aportacions originals d'aquest treball són el desenvolupament d'una nova formulació de les equacions dinàmiques de contacte, la definició de l'operador d'evolució K per a sistemes de contacte singulars, i un tractament innovador de les estructures precontacte. Per tal d'il·lustrar la teoria, proporcionem exemples de sistemes de contacte singulars basats en versions modificades de les lagrangianes de la partícula conforme i de Cawley.

Keywords

contact structure, Lagrangian mechanics, Hamiltonian mechanics, singular Lagrangian, implicit system, constraint algorithm, precontact structure

MSC

70G45; 37J39, 37J55, 70H03, 70H05, 70H45

Contents

1. Introduction

Geometric mechanics

Geometric mechanics is a modern approach to ordinary mechanics that employs the language and tools of differential geometry to describe and analyse physical systems. The use of mathematical structures such as tangent and cotangent bundles, flows of vector fields, differential forms, tensor fields, connections, Riemannian and pseudo-Riemannian geometry, as well as Lie groups and their actions on manifolds, establishes a correspondence between physical concepts and geometric objects. This perspective not only provides a coordinate-free and more unified formulation of ordinary mechanics but also offers powerful methods for studying dynamical systems with symmetries and conservation laws. These topics are covered in a number of books, for instance [1, 4, 40, 59].

The foundational ideas behind geometrical mechanics were developed in the 18th and 19th centuries, thanks to the works and discoveries of L. Euler, J.L. Lagrange, W.R. Hamilton, C.G.J. Jacobi, and G. Darboux, among others. Through their efforts, subjects such as celestial mechanics, rigid body motion, and variational principles were formulated in a precise mathematical language. It was in the second half of the 20th century that mathematicians such as S. Smale, G. Reeb, J.M. Souriau, V.I. Arnold, J. Moser, and J. Marsden, among many others, began to use the methods of differential geometry to study physical systems.

Since then, geometric mechanics has influenced numerous areas of mathematics by introducing geometric structures and methods rooted in physical intuition. Its development has shaped and enriched fields such as symplectic and Poisson geometry, differential topology, and the theory of Lie groups and Lie algebras. The geometric perspective on ordinary mechanics has provided a powerful theoretical framework applicable to the modern formulations of Lagrangian and Hamiltonian dynamics, control theory, integrable systems, and the study of more general dynamical systems.

Singular Lagrangians

It is well known that many of the dynamical systems that appear in ordinary mechanics admit two equivalent descriptions: the Lagrangian and the Hamiltonian formalisms. In the first one, the dynamics are described by the so-called *Euler–Lagrange equations*, a system of second-order differential equations in the configuration space. In the second one, the motion is defined by *Hamilton's equations*, which form a system of first-order differential equations in the phase space. Both formalisms are related by the so-called Legendre transformation.

The existence of this equivalence between the two formalisms is mainly due to the fact that the Lagrangian and Hamiltonian approaches are often used to represent Newtonian mechanics. One of the main ideas of Newtonian mechanics is the *principle of determinism*, that is, the complete state of a physical system at a given time *uniquely* determines its future (and past) evolution. This is mathematically justified because Newton's second law

$$\ddot{x}^i = f^i(t, x, \dot{x})$$

is a system of second-order differential equations in normal form, to which the theorem of existence and uniqueness of solutions applies.

However, the Lagrangian formalism, where the dynamics are determined by the paths that extremize the action functional $\int L \, \mathrm{d}t$, with L a function depending on positions and velocities, is broad enough to describe many dynamical systems that do not necessarily arise from a Newtonian framework. The problem of finding a Lagrangian function whose Euler–Lagrange equations reproduce a given dynamical system is known as the *inverse problem*. For instance, Helmholtz conditions [20] provide necessary and sufficient conditions for the existence of an appropriate Lagrangian function.

In this more general context, not every Lagrangian function determines a system of second-order differential equations that can be written in normal form. In such cases, existence and uniqueness of solutions are not guaranteed, and we say that the Lagrangian is *singular*. These systems are also characterized by the fact that the Legendre transformation is not a local diffeomorphism, which complicates the construction of an associated Hamiltonian formalism.

In the 1950s, P.A.M. Dirac [29, 30] and P. Bergmann [3, 7] independently developed a *constraint algorithm* that allows the dynamics of singular systems to be well-defined, under certain weakened regularity conditions. The aim of the constraint algorithm is twofold:

- One seeks to find the set of points in the velocity or phase space where solutions to the equations of motion exist. Typically, some regularity conditions are assumed at each step of the algorithm so that the dynamics evolve on a submanifold, known as the *final constraint submanifold*. This subset is defined by the vanishing of a collection of functions, referred to as the *constraints* of the system.
- One also aims to characterize the multiplicity of the solutions, which is described by the so-called *gauge transformations*. This non-uniqueness is a key feature of many physical theories, as gauge symmetries often reflect fundamental physical invariances, such as those appearing in electromagnetism, Yang–Mills theory, and general relativity.

With the development of differential-geometric methods for the study of dynamical systems, an intrinsic description of the Lagrangian and Hamiltonian formalisms was studied. After analysing the regular case, a *geometric* theory for singular Lagrangian functions was also developed. A geometrized version of the constraint algorithm for the more general *presymplectic systems* was first described by Gotay, Nester and Hinds in [43], which they later applied to Lagrangian systems in [41, 42]. After this, a lot of work was done to establish well the relation between the Lagrangian and the Hamiltonian formalism in the singular case [5, 6, 64, 65], as well as to develop constraint algorithms which can be applied to more general singular differential equations [21, 27, 49, 52, 54].

In particular, in this master thesis we study a powerful tool to deal with singular Lagrangians, the so-called K evolution operator. This geometrical object was first presented in coordinates by C. Batlle, J. Gomis, J.M. Pons, and N. Roman-Roy in [6], developing some ideas originally appearing in the work of K. Kamimura [60]. The K operator was first given an intrinsic, geometric characterization by X. Gràcia and J.M. Pons in [51] and by J.F. Cariñena and C. López in [16], in two different but equivalent ways. This operator is a vector field along the Legendre map, and it establishes an unambiguous and well-defined time evolution in the Lagrangian formulation of functions defined in the phase space. Also, it gives a geometric description of the Euler–Lagrange equations (which includes the second-order condition), and has properties relating the Hamiltonian constraint functions with the Lagrangian ones.

The K operator has been used to study gauge symmetries and other geometric structures along the Legendre map [50, 53]. It has also been applied to Hamilton–Jacobi theory [15], to the study of Lagrangian systems whose Legendre map has generic singularities [66, 67], and has been generalized to other formalisms such as time-dependent Lagrangian mechanics [13, 17], field theories [31, 70], higher-order Lagrangians [18, 55], or supermechanics [14]. The K evolution-operator is a particular example of a section along a map, to know more about these geometrical objects and their applications to physics see [11, 12].

Contact mechanics

In the late 19th century, Sophus Lie introduced the concept of *Berührungstransformationen* (contact transformations), a class of transformations that preserve the solution sets of certain differential equations. His work laid the groundwork for the modern theory of *contact geometry*, which has since found applications in a wide range of fields, including geometric optics, thermodynamics, Hamiltonian dynamics, and fluid mechanics. For a detailed historical account of the development of contact geometry and topology, as well as a thorough bibliography, see [38].

A *contact structure* on a $(2n+1)$-dimensional manifold is a maximally non-integrable hyperplane field. Locally, this hyperplane field can be expressed as the kernel of a 1-form α satisfying the non-integrability condition

$$\alpha \wedge (d\alpha)^n \neq 0 \,,$$

at every point. This condition ensures that the distribution defined by $\operatorname{Ker}\alpha$ is as far from being integrable as possible, in the sense of the Frobenius theorem. Note that if f is any nowhere-vanishing smooth function, then the kernel of $f\alpha$ defines the same distribution as the kernel of α, so the contact structure depends only on the conformal class of α. In this thesis we usually consider that the contact structure is defined entirely by the kernel of a fixed 1-form, in the literature some authors refer to this as *co-oriented* contact structures [45, 46]. Contact geometry is often viewed as the odd-dimensional counterpart of symplectic geometry. Both theories are governed by distinct non-degeneracy conditions on differential forms and exhibit a strong local rigidity: Darboux's theorem guarantees that all contact structures are locally equivalent, just as in symplectic geometry.

In recent years, there has been growing interest in developing geometric frameworks to model dissipative systems [10, 32, 63, 69]. In this context, the tools and structures of contact geometry have proven to be very effective (see for instance [8, 44, 68] for some applications). In particular, for mechanical systems, contact geometry has been successfully applied to describe non-conservative Lagrangian and Hamiltonian dynamics [25, 35]. This approach is analogous to the use of symplectic and cosymplectic manifolds as the natural geometric settings for autonomous and non-autonomous Hamiltonian systems, respectively. The first Lagrangian formulation of contact systems, as a variational principle, is due to G. Herglotz (see [39, 57, 58]), around 1930.

This formalism has also been generalized to treat non-autonomous, dissipative mechanical systems through the use of cocontact structures [22, 37]. In addition, Herglotz's variational principle has been extended to field theories [36], and more general geometric frameworks such as multicontact [23, 24, 77] and k-contact structures [28, 33, 35], have been developed to give a geometric treatment to dissipative field theories.

Analogously to the ordinary case, this Lagrangian formulation of dissipative mechanics admits *singular* Lagrangian functions. The consequences of considering such functions are similar: the generalized Euler–Lagrange equations cannot be written in normal form, and therefore, existence and uniqueness of solutions are not guaranteed. Moreover, the associated Hamiltonian formalism is no longer in one-to-one correspondence with the Lagrangian side, and the underlying geometric framework may deviate from contact geometry, as the relevant structures can exhibit degeneracies. In [26, 61], singular contact Lagrangian functions and precontact systems were studied, and a constraint algorithm for contact Hamiltonian systems was developed. However, that work considers only the case in which the precontact structure admits a Reeb vector field—an assumption that is not always satisfied. A broader and more general definition of precontact manifold was later introduced in [46].

Goals of the thesis

This master's thesis focuses on the study of singular contact Lagrangian systems. The main goal is to identify which are the key geometric structures involved in the description of such systems, both in the Lagrangian and Hamiltonian formalisms. To achieve this, we analyse various properties of precontact structures (existence of Reeb and Liouville-type vector fields, the class of precontact forms) and define their associated dynamics. In addition, we define the evolution operator K in the context of contact mechanics and explore its role in relating the contact Lagrangian and Hamiltonian formulations.

Structure of the dissertation

Chapter 2 This chapter is devoted to the geometric formulation of ordinary mechanics. We begin by reviewing a few fundamental notions of symplectic geometry. The Lagrangian and Hamiltonian formalisms, and their relation, are presented within the framework of symplectic geometry. Some variational aspects of the Lagrangian formalism are also included. In the last two sections we study the theory of singular Lagrangians. First, in Section 2.4, we examine weaker sufficient conditions under which the Lagrangian energy is $\mathcal{F}L$-projectable. This allows for the development of a suitable Hamiltonian formalism even in the singular case. Lastly, in Section 2.5, we study to the K evolution operator, a geometric object that is useful to study the relation between the Lagrangian and Hamiltonian formalisms, especially in the singular case. We give an intrinsic characterization of the K operator and we state and prove its main properties.

Chapter 3 In this chapter we study contact systems. We start with a review of the essential concepts of contact geometry. We introduce the definition of a contact manifold and the Reeb vector field, and we present both the existence and uniqueness theorem for the Reeb vector field, as well as Darboux's theorem for contact manifolds. With this geometric foundation in place, we define contact Hamiltonian systems and the corresponding contact Hamiltonian equations, presenting them in several equivalent forms—one of which notably does not involve the Reeb vector field. The final sections of this chapter are devoted to the study of the contact Lagrangian formalism and its associated contact Hamiltonian formulation.

Chapter 4 We begin the chapter with a review of the notions of rank and class of a differential form. In Section 4.2, we define precontact manifolds and provide a characterization of them in terms of the class of the 1-form. We investigate the existence of Reeb vector fields in such structures and highlight the role of Liouville vector fields, which serve as natural counterparts when Reeb vector fields fail to exist. In the final section of the chapter, we define precontact Hamiltonian systems using the new contact Hamiltonian equations (3.3) to study dynamics on precontact manifolds. The chapter concludes with some novel results relating certain functions and their Lie derivatives with respect to the Reeb or Liouville vector fields to changes in the parity of the class of the defining 1-form.

Chapter 5 In this chapter, we study contact singular Lagrangian systems and the relevant geometric structures associated with them. The exposition follows the approach presented in the last two sections of Chapter 2. First, in Section 5.1, we examine almost-regular Lagrangian functions and introduce several structures. With the use of these structures we show that, even under weaker regularity conditions, the contact Lagrangian energy remains $\mathcal{F}L$-projectable. Then, in Section 5.2, we provide an intrinsic characterization of the evolution operator K for contact systems in two equivalent ways. We analyse the main properties of the contact evolution operator. In the last section we present two examples of singular Lagrangian systems: the conformal particle and the Cawley Lagrangians, with added dissipation terms.

Chapter 6 This chapter is devoted to conclusions and outlook.

Appendix A In this appendix, we review some basic notions of vector bundles. We define the pull-back of a vector bundle and the vertical bundle, and we provide intrinsic definitions of several geometric objects that naturally arise in the Lagrangian formulations of both ordinary and contact mechanics.

The main references used for this master's thesis are the book *Géométrie différentielle et mécanique analytique* by C. Godbillon [40] and the articles "Theory of singular Lagrangians" by J.F. Cariñena [9], "On an evolution operator connecting Lagrangian and Hamiltonian formalisms" by X. Gràcia and J.M. Pons [51] and "New contributions to the Hamiltonian and Lagrangian contact formalisms for dissipative mechanical systems and their symmetries" by J. Gaset, X. Gràcia, M.C. Muñoz-Lecanda, X. Rivas and N. Román-Roy [35]. More specific references can be found at the beginning of each chapter and throughout the text.

Original contributions

This thesis makes several original contributions to the field of contact mechanics, mainly to the geometry of precontact manifolds and the formulation of singular contact Lagrangian systems.

We introduce two new, equivalent formulations of the contact Hamiltonian equations, namely Equations (3.3) and (3.4). These formulations do not depend on the Reeb vector field and are thus particularly well-suited for cases in which the Reeb field is not well-defined.

A substantial part of this work is devoted to the study and characterization of precontact manifolds. We investigate the conditions under which Reeb vector fields exist and highlight the important role played by Liouville vector fields in situations where Reeb fields are absent (Section 4.2). Using the newly introduced contact Hamiltonian equations, we define and analyse dynamics on precontact manifolds. We also show that conformal rescaling can change the class of the 1-form defining the precontact structure, and we provide conditions under which this occurs (Section 4.3).

In Section 5.2, we provide the first complete characterization of the K-evolution operator in the contact setting. Our results show that the main properties of this operator in the ordinary setting can be extended to the contact case.

2. Lagrangian and Hamiltonian mechanics

In this chapter, we review the geometric formulation of classical mechanics in order to establish the notation and main results, which will be generalized to the setting of dissipative mechanics in the following chapters. We pay special attention to the theory of singular Lagrangian systems and to the properties and characterization of the evolution operator K.

The main references used for the development of this chapter are [1, 4, 9, 51, 54].

2.1 Hamiltonian systems

Definition 2.1.1. Given a manifold M, a closed and non-degenerate 2-form $\omega \in \Omega^2(M)$ is called a *symplectic form*. We say that the pair (M, ω) is a *symplectic manifold*.

The non-degeneracy of ω means that, for each $p \in M$, the linear map $\widehat{\omega}_p \colon \mathrm{T}_p M \to \mathrm{T}^*_p M$, defined by $\widehat{\omega}_p(u_p) = \omega_p(u_p, \cdot)$, is an isomorphism. This non-degeneracy condition implies that the manifold M has even dimension. The condition of ω being closed means that $\mathrm{d}\omega = 0$.

Let (M, ω) be a symplectic manifold, and let $H \in \mathscr{C}^\infty(M)$ be a function on M, then, we say that the triple (M, ω, H) is a *Hamiltonian system*, with Hamiltonian function H. Any Hamiltonian system (M, ω, H), defines a unique *Hamiltonian vector field* $X_H \in \mathfrak{X}(M)$, with the so-called *Hamilton's equation*

$$i_{X_H}\omega = -\mathrm{d}H\,,$$

or $X_H = \widehat{\omega}^{-1} \circ \mathrm{d}H$.

We say that a vector field is *Hamiltonian* if it is the Hamiltonian vector field of a certain function. Additionally, a vector field X is *locally Hamiltonian* if, for every point $p \in M$, there exists a neighbourhood where X is Hamiltonian.

Proposition 2.1.2. *Given $X \in \mathfrak{X}(M)$, the following statements are equivalent:*

1. *A vector field X is locally Hamiltonian.*
2. *The 1-form $i_X\omega$ is closed.*
3. *We have $\mathscr{L}_X\omega = 0$.*

Remark 2.1.3 (Darboux coordinates)**.** By Darboux's theorem, given a symplectic manifold (M, ω) then, for each point $p \in M$, there exists a chart $(U, (x^i, p_i))$ in p such that

$$\omega|_U = \mathrm{d}x^i \wedge \mathrm{d}p_i\,.$$

The coordinates defined by this chart are the so-called *Darboux* or *symplectic* coordinates.

The integral curves of X_H are determined by *Hamilton's equations*

$$\dot{\xi} = X_H \circ \xi\,,$$

where $\xi \colon I \to M$ is a path on M, and $\dot{\xi}$ its canonical lift to $\mathrm{T}Q$. Their local expression, in Darboux coordinates (x^i, p_i), is

$$\frac{\mathrm{d}x^i}{\mathrm{d}t} = \frac{\partial H}{\partial p_i}\,, \quad \frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{\partial H}{\partial x^i}\,.$$

2.2 Lagrangian formalism

This section is divided into two parts. First, we provide a brief overview on how to obtain the Euler–Lagrange equations via variational calculus. Second, we give a geometric formulation of Lagrangian mechanics in terms of (pre)symplectic manifolds.

Let us consider a manifold Q, its tangent bundle $\mathrm{T}Q$, and a smooth function $L\colon \mathrm{T}Q \to \mathbb{R}$, called the *Lagrangian*. Given two points $q_1, q_2 \in Q$ and an interval $I = [t_1, t_2]$, we define the path space as

$$\Omega := \{\gamma\colon I \to Q \mid \gamma \text{ is a } \mathscr{C}^2 \text{ path such that } \gamma(t_1) = q_1, \gamma(t_2) = q_2\}\,,$$

and the *second-order submanifold* of $\mathrm{T}(\mathrm{T}Q)$ as

$$N(\mathrm{T}Q) \equiv \{w \in \mathrm{T}(\mathrm{T}Q) \mid \mathrm{T}\pi_Q(w) = \pi_{\mathrm{T}Q}(w)\} \subset \mathrm{T}(\mathrm{T}Q)\,,$$

where $\pi_{\mathrm{T}Q}\colon \mathrm{T}(\mathrm{T}Q) \to \mathrm{T}Q$ and $\pi_Q\colon \mathrm{T}Q \to Q$ are the canonical projections. The elements of $N(\mathrm{T}Q)$ are, in coordinates, of the form $(q, v; v, a)$.

We define the *action map* $S[\gamma]\colon \Omega \to \mathbb{R}$ as

$$S[\gamma] := \int_{t_1}^{t_2} L(\dot\gamma(t))\,\mathrm{d}t\,, \tag{2.1}$$

and a *variation* of a path γ as a $\mathscr{C}^2$ map $\Gamma\colon J \times I \to Q$, where $J \subset \mathbb{R}$ is open and contains 0. For every fixed $\epsilon \in J$, the map must satisfy that $\Gamma_\epsilon(t) = \Gamma(\epsilon, t) \in \Omega$ and $\Gamma_0(t) = \gamma(t)$. We denote by Γ' and $\dot\Gamma$ its lifts to $\mathrm{T}Q$ with respects to ϵ and to t, respectively. Also, we define the *infinitesimal variation* of Γ by $\mathbf{w}(t) = \Gamma'(0, t)$.

Definition 2.2.1 (Hamilton's principle)**.** We say that a path γ is a *critical path* of the action S if $\frac{\mathrm{d}}{\mathrm{d}\epsilon} S[\Gamma_\epsilon]\Big|_{\epsilon=0} = 0$ for all variations Γ of γ.

Theorem 2.2.2. *Considering the variational problem given by the action map* (2.1)*, we have*

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon} S[\Gamma_\epsilon]\Big|_{\epsilon=0} = \int_{t_1}^{t_2} \langle \mathrm{d}L(\dot\gamma(t)), \mathbf{w}^{(1)}(t)\rangle\,\mathrm{d}t\,,$$

where $\mathbf{w}^{(1)}(t) = \kappa_Q \circ \dot{\mathbf{w}}$, with κ_Q being the canonical involution of $\mathrm{TT}Q$.

Definition 2.2.3. The *Legendre transformation* of a Lagrangian function $L\colon \mathrm{T}Q \to \mathbb{R}$ is its fibre derivative $\mathcal{F}L\colon \mathrm{T}Q \to \mathrm{T}^*Q$, defined by

$$\langle \mathcal{F}L(u_q), v_q\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} L(u_q + t v_q)\,.$$

In natural coordinates

$$\mathcal{F}L(q^i, v^i) = \left(q^i, \frac{\partial L}{\partial v^i}\right)\,.$$

Definition 2.2.4. The 1-form $\mathcal{E}_L\colon N(\mathrm{T}Q) \to \mathrm{T}^*Q$, defined by

$$\langle \mathcal{E}_L \circ \ddot\gamma, \mathbf{w}\rangle = \langle \mathrm{d}L \circ \dot\gamma, \mathbf{w}^{(1)}\rangle - \mathrm{D}\langle \mathcal{F}L \circ \dot\gamma, \mathbf{w}\rangle\,. \tag{2.2}$$

is called the *Euler–Lagrange operator* associated with L. Locally, it is expressed as

$$\mathcal{E}_L = \left(\frac{\partial L}{\partial q^i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial v^i}\right)\right)\mathrm{d}q^i\,.$$

Theorem 2.2.5. *A path $\gamma\colon I \to Q$ is a critical path if and only if it satisfies the Euler–Lagrange equation*

$$\mathcal{E}_L \circ \ddot{\gamma} = 0\,,$$

which in natural coordinates is

$$\frac{\partial L}{\partial q^i}(\dot{\gamma}(t)) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial v^i}(\dot{\gamma}(t))\right) = 0\,.$$

For the second part of this section, we present the Lagrangian formulation written in terms of Hamiltonian systems and symplectic manifolds, as seen in Section 2.1. In other words, we intend to give $\mathrm{T}Q$ a symplectic structure, such that we can retrieve the Euler–Lagrange equations as the Hamilton equations defined by this symplectic structure and a suitable energy function. We will see that this is possible only under some regularity conditions. If these conditions are not satisfied we will need to use other techniques, which are presented and studied in the last two sections of this chapter.

Definition 2.2.6. The *Lagrangian 1-form* $\theta_L \in \Omega^1(\mathrm{T}Q)$, is the 1-form defined as

$$\theta_L = {}^t J \circ \mathrm{d}L\,,$$

where $J\colon \mathrm{TT}Q \to \mathrm{TT}Q$ is the so-called *vertical endomorphism* of the tangent bundle (see Appendix A), expressed in coordinates as $J = \frac{\partial}{\partial v^i} \otimes \mathrm{d}q^i$.

In natural coordinates, θ_L is written as

$$\theta_L = \frac{\partial L}{\partial v^i}\mathrm{d}q^i\,.$$

The *Lagrangian 2-form* $\omega_L \in \Omega^2(\mathrm{T}Q)$, is defined as $\omega_L = -d\theta_L$. It has local expression

$$\omega_L = \frac{\partial^2 L}{\partial v^i \partial q^j}\mathrm{d}q^i \wedge \mathrm{d}q^j + \frac{\partial^2 L}{\partial v^i \partial v^j}\mathrm{d}q^i \wedge \mathrm{d}v^j\,.$$

Definition 2.2.7. A Lagrangian function $L\colon \mathrm{T}Q \to \mathbb{R}$ is *regular* if the following equivalent conditions hold:

1. The Legendre transformation $\mathcal{F}L\colon \mathrm{T}Q \to \mathrm{T}^*Q$ is a local diffeomorphism.
2. At each $u_q \in \mathrm{T}Q$ the fibre Hessian $\mathcal{F}^2 L(u_q)$ is a non-degenerate bilinear form.
3. In natural coordinates the Hessian matrix $\left(\dfrac{\partial^2 L}{\partial v^i \partial v^j}\right)$ is everywhere non-singular.
4. The associated Lagrangian 2-form ω_L is a symplectic form.

The proof of these equivalences comes directly from analysing the local expressions of the objects involved.

The last step to construct the desired structure is to find an according function, which plays the role of the *energy*. For this, we first need to introduce a canonical object defined on $\mathrm{T}Q$, the so-called *Liouville vector field* $\Delta \in \mathfrak{X}(\mathrm{T}Q)$. Locally it is expressed as $\Delta = v^i \frac{\partial}{\partial v^i}$. With it we can construct the *Lagrangian energy function*

$$E_L = \Delta(L) - L\,.$$

An intrinsic characterization of the Liouville vector field is provided in Appendix A.

Definition 2.2.8 (Lagrangian system)**.** Given a manifold Q and a *Lagrangian function* $L\colon \mathrm{T}Q \to \mathbb{R}$, we say that $(\mathrm{T}Q, \omega_L, E_L)$ is a *Lagrangian system*.

Note that, if L is regular then ω_L is a symplectic form, and so the Lagrangian system $(\mathrm{T}Q, \omega_L, E_L)$ is symplectic. The non-degeneracy of ω_L also ensures that there exists a unique vector field X_L, the *Lagrangian vector field*, satisfying

$$i_{X_L}\omega_L = \mathrm{d}E_L \,. \tag{2.3}$$

With this construction, it is clear that the flow of X_L preserves the symplectic form ω_L, and that the energy E_L is also invariant along the integral curves of the Lagrangian vector field. Actually, even if the Lagrangian is not regular, if we can find a (not necessarily unique) vector field satisfying (2.3), these last two conditions hold, as they only depend on the fact that ω_L is closed.

Definition 2.2.9. We say that a vector field $X \in \mathfrak{X}(\mathrm{T}Q)$ is a *second-order vector field* (or a SODE) if it satisfies $J \circ X = \Delta$. This means that its local expression is

$$X = v^i \frac{\partial}{\partial q^i} + f^i \frac{\partial}{\partial v^i} \,,$$

and so, in coordinates, it defines a system of differential equations of the form

$$\frac{\mathrm{d}^2 q^i}{\mathrm{d}t^2} = f^i(q, \dot{q}) \,.$$

Proposition 2.2.10. *If the Lagrangian function L is regular, then the Lagrangian vector field $X_L\colon \mathrm{T}Q \to \mathrm{T}(\mathrm{T}Q)$ is a second-order vector field on $\mathrm{T}Q$ satisfying*

$$\mathcal{E}_L \circ X_L = 0 \,,$$

where $\mathcal{E}_L$ is the Euler–Lagrange operator, as defined in Equation (2.2)*. Thus, being a solution of the Euler–Lagrange equations is equivalent to being an integral curve of X_L.*

If the Lagrangian is regular, Equation (2.3) is enough to ensure that the Lagrangian vector field is a second order vector field. Otherwise, a vector field X (if it exists), which solves (2.3), is not necessarily of second-order.

The Euler–Lagrange equations are always well-defined, that is, they do not depend on the regularity of the Lagrangian function $L \in \mathscr{C}^\infty(\mathrm{T}Q)$. If one imposes (2.3) and the second-order condition, then the vector fields which satisfy these two conditions still have as integral curves the solutions of the Euler–Lagrange equations. Regularity provides existence and unicity of solutions, plus the advantage of being able to give the Lagrangian system a symplectic structure. If one does not have a regular Lagrangian, the Lagrangian 2-form ω_L is not symplectic, because it is *degenerate*. Closed 2-forms of constant rank receive the name of *presymplectic forms* [41, 42, 43, 64].

The consequences of dealing with non-regular Lagrangian systems are discussed further in the last two sections of this chapter.

2.3 Hamiltonian formalism

In the last section, we worked with a manifold Q, the configuration space, its tangent bundle $\mathrm{T}Q$ and a Lagrangian function $L\colon \mathrm{T}Q \to \mathbb{R}$, with which we constructed the Lagrangian system $(\mathrm{T}Q, \omega_L, E_L)$.

Here, we relate this system with the *canonical symplectic manifold* on the phase space $(\mathrm{T}^*Q, \omega_Q)$. This is done using the Legendre transformation $\mathcal{F}L\colon \mathrm{T}Q \to \mathrm{T}^*Q$, defined in 2.2.3.

Note that, for any manifold Q, its cotangent bundle T^*Q has a canonical symplectic structure. Namely, the 2-form $\omega_Q \in \Omega^2(\mathrm{T}^*Q)$, defined as $\omega_Q = -\mathrm{d}\theta_Q$, is a symplectic form, where the 1-form $\theta_Q \in \Omega^1(\mathrm{T}^*Q)$ is given by

$$\theta_Q(\alpha_p) = {}^t(\mathrm{T}_p\pi_Q) \cdot \alpha_p \,.$$

Here, $\alpha_p \in \mathrm{T}^*_pQ$ and $\pi_Q\colon \mathrm{T}^*Q \to Q$ is the canonical projection.

Therefore, the cotangent bundle of any given manifold Q is canonically a symplectic manifold with $(\mathrm{T}^*Q, \omega_Q)$.

Theorem 2.3.1. *Given a Lagrangian function $L \in \mathscr{C}^\infty(\mathrm{T}Q)$, one has*

$$\theta_L = \mathcal{F}L^*(\theta_Q)\,, \qquad \omega_L = \mathcal{F}L^*(\omega_Q)\,.$$

Here, θ_Q and ω_Q are, respectively, the canonical 1-form and 2-form, while θ_L and ω_L are the Lagrangian 1-form and 2-form.

Definition 2.3.2. A Lagrangian function $L\colon \mathrm{T}Q \to \mathbb{R}$ is *hyperregular* if its Legendre transformation $\mathcal{F}L\colon \mathrm{T}Q \to \mathrm{T}^*Q$ is a global diffeomorphism.

Given a hyperregular Lagrangian function L, with the energy function E_L we can define the *Hamiltonian function* as $H = E_L \circ (\mathcal{F}L)^{-1}$. This gives rise to the Hamiltonian system in the phase space $(\mathrm{T}^*Q, \omega_Q, H)$. With this construction, the corresponding Hamiltonian vector field X_H is also $\mathcal{F}L$-related to the Lagrangian vector field X_L, i.e. $(\mathcal{F}L)_* X_L = X_H$. As a direct consequence, their integral curves are also $\mathcal{F}L$-related.

If the Legendre transformation is not a global diffeomorphism, we still have this one-to-one correspondence between the Lagrangian and Hamiltonian formalisms at least locally.

Note that, if the Lagrangian function is *not* regular then we cannot construct a related Hamiltonian formalism in this way. However, even in this case, one always has that the integral paths of the Hamiltonian formalism need to satisfy the conditions

$$\begin{cases} p = \dfrac{\partial L}{\partial v}\,, \\ \dot{p} = \dfrac{\partial L}{\partial q}\,, \end{cases} \tag{2.4}$$

if we want the integral paths of the two formalisms to be $\mathcal{F}L$-related. This is just a consequence of the Euler–Lagrange equations. Equations (2.4) are the so-called Hamilton–Dirac equations for the Lagrangian function L.

2.4 Singular Lagrangians

For the last two sections, we have mostly given results for regular Lagrangian functions. This gave the Lagrangian 2-form ω_L non-degeneracy, which allowed us to construct a symplectic system $(\mathrm{T}Q, \omega_L, E_L)$. With this construction, the integral curves of the associated Lagrangian vector field X_L were exactly the solutions of the Euler–Lagrange equation. Under these regularity conditions, we additionally obtained a Hamiltonian system $(\mathrm{T}^*Q, \omega_Q, H)$ in the phase space, which is $\mathcal{F}L$-related to the Lagrangian system. In this way, the Lagrangian and the Hamiltonian formulations are put in a one-to-one correspondence.

Therefore, the natural question that arises now is what happens when the Lagrangian function is *singular*, i.e. is not regular. In this case, the Euler–Lagrange equations (2.2.5) cannot be written in normal form, because the matrix of the coefficients of the accelerations is not invertible. Therefore, we are not able to apply the theorem for the existence and uniqueness of solutions of systems of differential equations. This implies that the equations of motion do not necessarily admit solutions for every initial condition; and when solutions do exist, they lack uniqueness. This contradicts the principle of determinism in classical physics, as there are degrees of freedom in the solutions—specifically, the gauge degrees of freedom.

In this section, we consider only singular Lagrangian functions. However, we still impose certain weaker regularity conditions that allow us to construct a suitable Hamiltonian formalism.

We say that a Lagrangian function L is *almost-regular* [41, 42] if:

- The image of the Legendre map $P_0 := \mathcal{F}L(\mathrm{T}Q) \subset \mathrm{T}^*Q$ is a closed submanifold, called the *primary Hamiltonian constraint submanifold*,
- The induced map $\mathcal{F}L_0 \colon \mathrm{T}Q \to P_0$ is a submersion and has connected fibres.

These technicals conditions are the most basic requirements necessary to develop a Hamiltonian formulation from a singular Lagrangian L, as we will see in the following exposition. From a local point of view it is sufficient to assume that $\mathcal{F}L$ has constant rank.

As we assume $P_0 := \mathcal{F}L(\mathrm{T}Q)$ to be a closed submanifold, it is locally defined by the vanishing of an independent set of functions $\{\phi_\mu\}_{\mu=1,\dots,m}$ with linearly independent differentials $\mathrm{d}\phi_\mu$, at every point of P_0. These functions are the so-called *primary Hamiltonian constraint functions*.

Now, consider the following lemma:

Lemma 2.4.1. *Let $\alpha \colon N \to P$ be a submersion at $y \in N$. Then, there exists an open neighbourhood $V \subset N$ of y such that $\alpha(V) \subset P$ is a submanifold. Additionally, we have:*

1. $\mathrm{Im}\,\mathrm{T}_y(\alpha) = \mathrm{T}_{\alpha(y)}(\alpha(V))$ *.*
2. *If $\alpha(V) \subset P$ is the submanifold defined locally by the vanishing of a set of functions ϕ_μ, with differentials $\mathrm{d}\phi_\mu$ linearly independent at each point, then $\mathrm{Ker}\,{}^t\mathrm{T}_y\alpha$ admits $\mathrm{d}_{\alpha(y)}\phi_\mu$ as a basis, and $\mathrm{Ker}\,{}^t\mathrm{T}\alpha$ admits $\mathrm{d}\phi_\mu \circ \alpha$ as a reference over V.*

Therefore, the $\mathrm{d}\phi_\mu \circ \mathcal{F}L$ form a (local) reference of the vector subbundle $\mathrm{Ker}\,{}^t\mathrm{T}(\mathcal{F}L) \subset \mathrm{T}^*\mathrm{T}^*Q$. Also, if we consider the canonical 2-form ω_Q in T^*Q and the functions ϕ_μ, we can define their corresponding (unique) Hamiltonian vector fields

$$X_\mu = X_{\phi_\mu} = \widehat{\omega}_Q^{-1} \circ \mathrm{d}\phi_\mu \,,$$

and from these we can write $X_\mu \circ \mathcal{F}L$, which constitute a reference for the vector subbundle $\widehat{\omega}_Q^{-1}(\mathrm{Ker}\,{}^t\mathrm{T}(\mathcal{F}L)) = \mathrm{Ker}\,({}^t\mathrm{T}(\mathcal{F}L) \circ \widehat{\omega}_Q) \subset \mathrm{T}Q \times_{\mathcal{F}L} \mathrm{T}\mathrm{T}^*Q$.

Now, recall also the following lemma:

Lemma 2.4.2. *Let $P_0 \overset{j}{\hookrightarrow} P$ be a submanifold defined by the vanishing of $\{\phi_\mu\}_{\mu=1,\dots,m}$, such that their differentials $\mathrm{d}\phi_\mu$ are linearly independent at every point of P_0. Then, $\mathrm{Ker}\,{}^t\mathrm{T}(j)$ admits $\{\mathrm{d}\phi_\mu|_{P_0}\}_{\mu=1,\dots,r}$ as a reference. Also, at every point $x \in P_0$ a tangent vector $v_x \in \mathrm{T}_xP$ is in T_xP_0 if and only if, for every ϕ_μ, it satisfies $\mathrm{d}\phi_\mu(v_x) = 0$.*

We can apply this lemma to $P_0 \overset{j}{\hookrightarrow} \mathrm{T}^*Q$, so that the $\mathrm{d}\phi_\mu|_{P_0}$ form, at least locally, a reference for the vector subbundle $\mathrm{Ker}\, {}^t\mathrm{T}(j) \subset P_0 \times_j \mathrm{T}^*\mathrm{T}^*Q$, and the $X_\mu|_{P_0}$ form a reference for the subbundle $F_0 = \mathrm{Ker}\,({}^t\mathrm{T}(j) \circ \widehat{\omega}_Q|_{P_0}) \subset P_0 \times_j \mathrm{TT}^*Q$.

On the other hand, the assumption that $\mathcal{F}L$ is a submersion is equivalent to saying that $\mathrm{Ker}\,\mathrm{T}(\mathcal{F}L)$ is a vector subbundle of $\mathrm{TT}Q$. Actually, as $\mathrm{T}(\tau_Q) = \mathrm{T}(\pi_Q) \circ \mathrm{T}(\mathcal{F}L)$, we have that $\mathrm{Ker}\,\mathrm{T}(\mathcal{F}L)$ is contained in the vertical subbundle $V(\mathrm{T}Q)$ of $\mathrm{T}(\mathrm{T}Q)$.

As the rank of $\mathrm{Ker}\,\mathrm{T}(\mathcal{F}L)$ is constant, let us say equal to m, we can find a local reference generated by m sections Γ_μ of $\mathrm{T}Q$, linearly independent at every point. Such a local frame can be, in fact, constructed from the primary Hamiltonian constraint functions.

From any $X_\phi \circ \mathcal{F}L = \widehat{\omega}_Q^{-1} \circ \mathrm{d}\phi \circ \mathcal{F}L$, where $\phi \in \mathscr{C}^\infty(\mathrm{T}^*Q)$, we can define a vector field Γ_ϕ on $\mathrm{T}Q$, which has local coordinates

$$\Gamma_\phi = \mathcal{F}L^* \left(\frac{\partial \phi}{\partial p}\right) \frac{\partial}{\partial v}\,.$$

See Appendix A for a detailed intrinsic characterization of these vector fields.

The following result follows from such constructions.

Proposition 2.4.3. *The vector fields $\Gamma_\mu = \Gamma_{\phi_\mu}$, constructed from the primary Hamiltonian constraint functions ϕ_μ, form a local reference for* $\mathrm{Ker}\,\mathrm{T}(\mathcal{F}L)$. *Their local expression is*

$$\Gamma_\mu = \gamma_\mu \frac{\partial}{\partial v}\,,$$

where the functions

$$\gamma_\mu = \mathcal{F}L^* \left(\frac{\partial \phi_\mu}{\partial p}\right)$$

form a basis of the kernel of the Hessian matrix $W = \left(\dfrac{\partial^2 L}{\partial v^i \partial v^j}\right)$.

Proof. By the chain rule

$$\mathrm{d}\mathcal{F}L^*(\phi) = \left(\mathcal{F}L^* \left(\frac{\partial \phi}{\partial q}\right) + \mathcal{F}L^* \left(\frac{\partial \phi}{\partial p}\right) \frac{\partial^2 L}{\partial v \partial q}\right) \mathrm{d}q + \mathcal{F}L^* \left(\frac{\partial \phi}{\partial p}\right) W \mathrm{d}v\,,$$

and if ϕ_μ is a primary Hamiltonian constraint function then $\mathrm{d}\mathcal{F}L^*(\phi_\mu) = 0$ necessarily, which implies that $\gamma_\mu W = 0$. Additionally, locally we check that

$$\mathrm{T}(\mathcal{F}L) \circ \Gamma_\mu(q, v) = \left(q, \frac{\partial L}{\partial v}, 0, \gamma_\mu W\right)\,,$$

and so, $\Gamma_\mu \in \mathrm{Ker}\,\mathrm{T}(\mathcal{F}L)$. Finally, note that the m vector fields are linearly independent because the γ_μ are, as a consequence of the linear independency of the $\mathrm{d}\phi_\mu$. Hence, the vector fields form a reference for $\mathrm{Ker}\,\mathrm{T}(\mathcal{F}L)$, and the γ_μ form a reference for $\mathrm{Ker}\, W$. □

With this, we can prove the most important result for the characterization of a Hamiltonian formalism in this context.

Proposition 2.4.4. *If the Legendre map $\mathcal{F}L$ is a submersion, then the Lagrangian energy function E_L is locally $\mathcal{F}L$-projectable, that is, there exists a function $H \in \mathscr{C}^\infty(\mathrm{T}^*Q)$ such that (locally) we have $E_L = H \circ \mathcal{F}L$.*

Proof. A necessary and sufficient condition for the existence of H is that $\Gamma_\mu(E_L) = 0$, i.e. if a function vanishes by $\operatorname{Ker} \mathrm{T}(\mathcal{F}L)$ then it is $\mathcal{F}L$-projectable. We have

$$\Gamma_\mu(E_L) = \gamma_\mu \frac{\partial}{\partial v}\left(v\frac{\partial L}{\partial v} - L\right) = \gamma_\mu(Wv) = 0\,,$$

and so the existence of H is guaranteed. □

If the Lagrangian function L is almost-regular, then E_L is *globally* $\mathcal{F}L$-projectable at P_0, that is, there exists a *unique* function $H_0\colon P_0 \to \mathbb{R}$, the *Hamiltonian* function, such that $\mathcal{F}L^*(H_0) = E_L$.

Once a Hamiltonian function on P_0 is defined, we can consider the presymplectic 2-form

$$\omega_0 = j^*(\omega_Q)\,,$$

to obtain the presymplectic Hamiltonian system (P_0, ω_0, H_0). The solutions and constraints that appear when solving

$$i_X\omega_0 = \mathrm{d}H_0\,,$$

are related with those of the presymplectic system $(\mathrm{T}Q, \omega_L, E_L)$. The relations between the two presymplectic systems can be studied by means of the evolution operator K, which is presented in the following section.

2.5 Evolution operator

Due to the lack of uniqueness of solutions —that is, if they exist—, the time-evolution operators for singular Lagrangian systems are generally not well-defined. These operators include terms corresponding to the (final) first-class primary Hamiltonian constraints, which are arbitrary.

In this context, however, there exists a time-evolution operator, usually denoted by K, which, when applied to functions on the cotangent bundle T^*Q, provides an unambiguous and well-defined time evolution in the Lagrangian formulation. In this section we characterize the evolution operator K, and we study its properties relating the Lagrangian and Hamiltonian formalisms.

2.5.1 Characterization of the evolution operator K

This operator was first introduced in coordinates in [6]. Later, it was given an intrinsic, geometric characterization in [51], which is the one we present here.

Its coordinate expression, in local coordinates, is

$$K(q, v) = v^i \left.\frac{\partial}{\partial q^i}\right|_{\mathcal{F}L(q,v)} + \frac{\partial L}{\partial q^i}\left.\frac{\partial}{\partial p_i}\right|_{\mathcal{F}L(q,v)}\,. \tag{2.5}$$

It can be shown that this expression is independent of the choice of local coordinates. Observe that, for any given Lagrangian L, this operator defines a map $K\colon \mathscr{C}^\infty(\mathrm{T}^*Q) \to \mathscr{C}^\infty(\mathrm{T}Q)$, given in natural coordinates as

$$(K \cdot f)(q, v) = v^i \mathcal{F}L^*\left(\frac{\partial f}{\partial q^i}\right) + \frac{\partial L}{\partial q^i}\mathcal{F}L^*\left(\frac{\partial f}{\partial p_i}\right)\,.$$

That is, given a function f on the cotangent bundle, the operator K gives the time derivative of the function in Lagrangian terms.

The key idea to understand the evolution operator K is to view it as a *section along a map*.

Definition 2.5.1. Let $\pi\colon E \to M$ be a vector bundle. Then, for another manifold N, and a given map $f\colon N \to M$ we say that $s\colon N \to E$ is a *section along* f if the following diagram is commutative:

$$\begin{tikzcd} & E \arrow[d, "\pi"] \\ N \arrow[ur, "s"] \arrow[r, "f"'] & M \end{tikzcd}$$

In the particular case where the vector bundle is $E = \mathrm{T}M$ or $E = \bigwedge^k \mathrm{T}^*M$, we call a section along f, respectively, a vector field or a k-form along f.

Example 2.5.2. The easiest and most well-known example of a section along a map is the case of the canonical lift of a path. Consider a manifold Q and its canonical tangent bundle $\mathrm{T}Q$. It is clear that $\tau_Q\colon \mathrm{T}Q \to Q$ is a vector bundle, with τ_Q the canonical projection. Then, for any path $\gamma\colon I \to Q$, the canonical lift $\dot\gamma\colon I \to \mathrm{T}Q$ is a section along the path γ.

With this concept in mind, we can see the evolution operator K as a vector field along the Legendre transformation $\mathcal{F}L$. Indeed, at each point $(q,v) \in \mathrm{T}Q$, the operator K derives functions on the cotangent bundle along vectors tangent to $\mathcal{F}L(q,v) \in \mathrm{T}^*Q$, that is, the evolution operator K is a map $K\colon \mathrm{T}Q \to \mathrm{T}(\mathrm{T}^*Q)$ such that

$$\tau_{\mathrm{T}^*Q} \circ K = \mathcal{F}L,$$

where τ_{T^*Q} is the canonical projection $\tau_{\mathrm{T}^*Q}\colon \mathrm{T}(\mathrm{T}^*Q) \to \mathrm{T}^*Q$. This last equation, in the literature, is sometimes referred to as the *structural condition* of the evolution operator K. For a function $f \in \mathscr{C}^\infty(\mathrm{T}^*Q)$, note that we can write $(K \cdot f) \in \mathscr{C}^\infty(\mathrm{T}Q)$ as the function defined by

$$(K \cdot f)(q,v) := \langle \mathrm{d}f(\mathcal{F}L(q,v)), K(q,v)\rangle \,. \tag{2.6}$$

Now, a vector field along the Legendre transformation $\mathcal{F}L$ has the general local expression

$$K(q,v) = \left(q, \frac{\partial L}{\partial v}, a(q,v), b(q,v)\right) ,$$

or, alternatively,

$$K(q,v) = a^i(q,v) \left.\frac{\partial}{\partial q^i}\right|_{\mathcal{F}L(q,v)} + b^i(q,v) \left.\frac{\partial}{\partial p_i}\right|_{\mathcal{F}L(q,v)} .$$

So, in order to obtain a fully intrinsic characterization of the evolution operator K (that is, to express it without the use of coordinates), we need conditions that determine $a^i(q,v)$ and $b^i(q,v)$, so that they correspond with (2.5). The next proposition geometrically characterizes the evolution operator K.

Proposition 2.5.3. *The evolution operator K is the only vector field along $\mathcal{F}L$ that satisfies the two following conditions:*

1. $\mathcal{F}L^*(i_K(\omega_Q \circ \mathcal{F}L)) = \mathrm{d}E_L$, *where $\omega_Q \in \Omega^2(\mathrm{T}^*Q)$ is the canonical 2-form in the cotangent bundle and E_L is the Lagrangian energy function.*

2. $\mathrm{T}(\pi_Q) \circ K = \mathrm{Id}_{\mathrm{T}Q}$, *with π_Q the canonical projection $\pi_Q\colon \mathrm{T}^*Q \to Q$.*

These two conditions are referred to in the literature as the *dynamical condition* and the *second-order condition*, respectively. That is because there is a clear analogy between these conditions and those satisfied by a second-order Lagrangian vector field. Recall that any vector field $X \in \mathfrak{X}(\mathrm{T}Q)$ satisfies

$$\tau_{\mathrm{T}Q} \circ X = \mathrm{Id}_{\mathrm{T}_Q},$$

with $\tau_{\mathrm{T}Q} \colon \mathrm{T}(\mathrm{T}Q) \to \mathrm{T}Q$ the canonical projection, by definition. Also, for a given Lagrangian L, the Lagrangian vector field X_L has to satisfy the so-called dynamical condition

$$i_{X_L}(\mathcal{F}L^*(\omega_Q)) = \mathrm{d}E_L .$$

And finally, for the vector field to be second-order one has to impose that

$$\mathrm{T}(\tau_Q) \circ X_L = \mathrm{Id}_{\mathrm{T}Q} .$$

If the Lagrangian is regular, by Proposition 2.2.10, the dynamical condition is enough to guarantee that it also satisfies the second-order condition. However, in the case we are interested in now, that is, when the Lagrangian is singular, the dynamical condition does not imply that the second-order condition is satisfied. The analogy between these 3 equations and the ones satisfied by the operator K is clear.

Let us now prove Proposition 2.5.3, by seeing that the two conditions do determine the operator K. Recall that, as K is a vector field along $\mathcal{F}L$, it has the local expression

$$K(q, v) = \left(q, \frac{\partial L}{\partial v}, a(q, v), b(q, v)\right),$$

where a^i, b^i are to be determined.

The second-order condition is

$$\mathrm{T}(\pi_Q) \circ K = \mathrm{Id}_{\mathrm{T}Q},$$

where $\pi_Q \colon \mathrm{T}^*Q \to Q$ is the canonical projection. The expression of $\mathrm{T}(\pi_Q) \colon \mathrm{T}(\mathrm{T}^*Q) \to \mathrm{T}Q$, in local coordinates, is

$$\mathrm{T}(\pi_Q)(q, p, v, u) = (q, v) .$$

And therefore,

$$\mathrm{T}(\pi_Q) \circ K(q, v) = (q, a(q, v)) .$$

So, by the second-order condition, we have

$$(q, a(q, v)) = \mathrm{Id}_{\mathrm{T}Q} = (q, v),$$

which implies that $a^i(q, v) = v^i$.

The dynamical condition is

$$\mathcal{F}L^*(i_K(\omega_Q \circ \mathcal{F}L)) = \mathrm{d}E_L ,$$

where ω_Q is the canonical 2-form on T^*Q, locally expressed as $\omega = \mathrm{d}q^i \wedge \mathrm{d}p_i$, and $E_L = \Delta(L) - L$ is the Lagrangian energy. On the one hand,

$$i_K(\omega_Q \circ \mathcal{F}L) = v^i \mathrm{d}p_i|_{\mathcal{F}L(q,v)} - b^i(q, v)\mathrm{d}q^i|_{\mathcal{F}L(q,v)},$$

and,

$$\mathcal{F}L^*(i_K(\omega_Q \circ \mathcal{F}L)) = \left(v^j \frac{\partial^2 L}{\partial v^j \partial q^i} - b^i(q, v)\right) \mathrm{d}q^i|_{\mathcal{F}L(q,v)} + v^i \frac{\partial^2 L}{\partial v^i \partial v^j} \mathrm{d}v^i|_{\mathcal{F}L(q,v)} .$$

Where last equation comes from the matrix of $\mathrm{T}(\mathcal{F}L)\colon \mathrm{T}(\mathrm{T}Q) \to \mathrm{T}(\mathrm{T}^*Q)$ being

$$\left(\begin{array}{c|c} \mathrm{Id} & 0 \\ \hline \frac{\partial^2 L}{\partial v^i \partial q^j} & \frac{\partial^2 L}{\partial v^i \partial v^j} \end{array}\right).$$

On the other hand, in local coordinates,

$$\mathrm{d}E_L = \left(v^j \frac{\partial^2 L}{\partial v^j \partial q^i} - \frac{\partial L}{\partial q^i} \right) \mathrm{d}q^i + v^i \frac{\partial^2 L}{\partial v^i \partial v^j} \mathrm{d}v^i \,.$$

And therefore, comparing terms, we necessarily have

$$b^i(q, v) = \frac{\partial L}{\partial q^i}\,,$$

Hence, the operator K is defined by these conditions, locally, as in (2.5), as we wanted to see.

Remark 2.5.4. There are other equivalent ways in which to intrinsically characterize the operator K.

- One way is through the so-called *Tulczyjew's triples* [75, 76], which is a formalism that uses the canonical diffeomorphisms

$$\begin{aligned} \alpha\colon \mathrm{T}(\mathrm{T}^*Q) &\longrightarrow \mathrm{T}^*(\mathrm{T}Q)\,, \\ \alpha(q, p, v, u) &\longmapsto (q, v, u, p) \end{aligned} \qquad \begin{aligned} \beta\colon \mathrm{T}(\mathrm{T}^*Q) &\longrightarrow \mathrm{T}^*(\mathrm{T}^*Q) \\ \beta(q, p, v, u) &\longmapsto (q, p, u, -v) \end{aligned}$$

  to obtain the following commutative diagram

$$\begin{tikzcd} \mathrm{T}^*(\mathrm{T}Q) \arrow[dr, "\pi_{\mathrm{T}Q}"'] & & \mathrm{T}(\mathrm{T}^*Q) \arrow[ll, "\alpha"'] \arrow[rr, "\beta"] \arrow[dl, "\mathrm{T}(\pi_Q)"] \arrow[dr, "\tau_{\mathrm{T}^*Q}"] & & \mathrm{T}^*(\mathrm{T}^*Q) \arrow[dl, "\pi_{\mathrm{T}^*Q}"] \\ & \mathrm{T}Q & & \mathrm{T}^*Q & \end{tikzcd}$$

  In this context, we can obtain the evolution operator as $K = \alpha^{-1} \circ \mathrm{d}L$.
- Another characterization is the one provided in [16], where the authors make use of the Skinner–Rusk formalism [73, 74], which mixes in a single description the Lagrangian and the Hamiltonian formalisms. In this context, the Whitney sum $W_0 = \mathrm{T}^*Q \oplus \mathrm{T}Q$ serves as the evolution space. This framework enables a unified treatment of both velocities and momenta. A key property of this formalism is that the dynamical condition in W_0 inherently incorporates the second-order condition, regardless of whether the Lagrangian is regular or not.

2.5.2 Properties relating the Lagrangian and Hamiltonian formalisms

Let us now see the importance of this evolution operator, by stating some of its most remarkable properties.

Proposition 2.5.5. *Let $\xi\colon I \to \mathrm{T}Q$ be a path in the tangent bundle and $\dot{\xi}\colon I \to \mathrm{T}(\mathrm{T}Q)$ its canonical lift. Then, ξ is a solution of the Euler–Lagrange equations, for a given Lagrangian L, if and only if*

$$\mathrm{T}(\mathcal{F}L) \circ \dot{\xi} = K \circ \xi\,. \tag{2.7}$$

Proof. The proof follows directly from the coordinate expressions. If $\xi = (q, v)$, then its canonical lift can be expressed as $\dot{\xi} = (q, v, \dot{q}, \dot{v})$. Thus, we have

$$\mathrm{T}(\mathcal{F}L) \circ \dot{\xi} = \left(q, \frac{\partial L}{\partial v}, \dot{q}, \dot{q} \frac{\partial^2 L}{\partial q \partial v} + \dot{v} \frac{\partial^2 L}{\partial v \partial v} \right),$$
$$K \circ \xi = \left(q, \frac{\partial L}{\partial v}, v, \frac{\partial L}{\partial q} \right),$$

and equating them, this is equivalent to

$$\dot{q} = v\,,$$
$$\dot{v} \frac{\partial^2 L}{\partial v \partial v} = \frac{\partial L}{\partial q} - \dot{q} \frac{\partial^2 L}{\partial q \partial v}\,,$$

which are precisely the Euler–Lagrange equations. □

Note that, if a path $\xi \colon I \to \mathrm{T}Q$ satisfies

$$\mathrm{T}(\mathcal{F}L) \circ \dot{\xi} = K \circ \xi$$

then, necessarily, it can be obtained as the canonical lift of a path $\zeta \colon I \to Q$, i.e. $\xi = \dot{\zeta}$. Therefore, Equation (2.7) implements the second-order condition, independently of the regularity of the Lagrangian.

A solution to the Euler–Lagrange equations satisfies $\dot{\xi} = X_L \circ \xi$, with X_L a second-order Lagrangian vector field defined on an appropriate submanifold of $\mathrm{T}Q$. Then, an immediate consequence of the last proposition is that we can write

$$\mathrm{T}(\mathcal{F}L) \circ X_L \circ \xi = K \circ \xi\,.$$

And note that, if S_f is the final constraint submanifold of $\mathrm{T}Q$, we have solutions at every point of the submanifold, and therefore

$$K|_{S_f} = \mathrm{T}(\mathcal{F}L) \circ X_L|_{S_f}\,.$$

Recall that if the Lagrangian is regular then the Legendre transformation is a local diffeomorphism, and the Lagrangian vector field X_L is unique. And so, we directly obtain the following equality

$$X_L = \mathrm{T}(\mathcal{F}L^{-1}) \circ K\,.$$

As we have seen, Proposition 2.5.5 shows the relation that exists between Lagrangian vector fields and the evolution operator K. The following proposition relates Hamiltonian vector fields and the evolution operator K.

Proposition 2.5.6. *Let $\psi \colon I \to \mathrm{T}^*Q$ be a path on the cotangent bundle, and $\dot{\psi} \colon I \to \mathrm{T}(\mathrm{T}^*Q)$ its canonical lift. Then, ψ is a solution to Hamilton–Dirac's equations for L if and only if*

$$\dot{\psi} = K \circ \mathrm{T}(\pi_Q) \circ \dot{\psi}\,.$$

Proof. Similarly to the last proposition, to prove this it is enough to explicitly write both sides of the equation in coordinates. If $\psi = (q, p)$, then its canonical lift is expressed as $\dot{\psi} = (q, p, \dot{q}, \dot{p})$. Therefore,

$$K \circ \mathrm{T}(\pi_Q) \circ \dot{\psi} = \left(q, \frac{\partial L}{\partial v}, \dot{q}, \frac{\partial L}{\partial q} \right),$$

and equating the left-hand side to the right-hand side yields

$$p = \frac{\partial L}{\partial v}, \qquad \dot{p} = \frac{\partial L}{\partial q},$$

which are precisely the Hamilton–Dirac equations for the Lagrangian L. □

Note that, if there exists a solution $\psi\colon I \to \mathrm{T}^*Q$ to the Hamilton–Dirac equations, then it can be expressed as

$$\psi = \mathcal{F}L \circ \xi$$

where $\xi\colon I \to \mathrm{T}Q$ is the canonical lift of the projected path $(\pi_Q \circ \psi)$. Then, using this and Proposition 2.5.5, we have

$$\dot{\psi} = \mathrm{T}(\mathcal{F}L) \circ \dot{\xi} = K \circ \xi = K \circ \mathrm{T}(\pi_Q) \circ \dot{\psi}\,.$$

Actually, the solutions to both formalisms are in bijection. Namely, the map $\xi \mapsto \mathcal{F}L \circ \psi$ sends solutions to solutions, with inverse $\psi \mapsto \mathrm{T}(\pi_Q) \circ \dot{\psi}$. This comes as a consequence of the map K being an embedding.

Now, suppose that we have a Hamiltonian vector field X_H, defined on the image of the Legendre transformation $\mathcal{F}L(\mathrm{T}Q) \subseteq \mathrm{T}^*Q$, for the Hamiltonian formalism associated to the Lagrangian system $(\mathrm{T}Q, L)$. Then, the solutions to Hamilton–Dirac's equations can be written as its integral curves $\dot{\psi} = X_H \circ \psi$, and we can write

$$K \circ \xi = \dot{\psi} = X_H \circ \psi = X_H \circ \mathcal{F}L \circ \xi\,. \tag{2.8}$$

This, in the final constraint submanifold S_f, gives the relation

$$K|_{S_f} = X_H \circ \mathcal{F}L|_{S_f}\,.$$

And, if the Lagrangian is regular, then we have

$$X_H = K \circ \mathcal{F}L^{-1}\,.$$

Lastly, assume that we have two solutions ξ and ψ of, respectively, the Euler–Lagrange equations and Hamilton's equations which are related, i.e. $\psi = \mathcal{F}L \circ \xi$ with ξ the canonical lift of the projected path $(\pi_Q \circ \psi)$. Then, for any given function $f \in \mathscr{C}^\infty(\mathrm{T}^*Q)$, the following holds:

$$\frac{\mathrm{d}}{\mathrm{d}t}(f \circ \psi) = \langle \mathrm{d}f \circ \psi, \dot{\psi}\rangle = \langle \mathrm{d}f \circ (\mathcal{F}L \circ \xi), K \circ \xi\rangle = (K \cdot f) \circ \xi\,,$$

where Equations (2.6) and (2.8) have been used.

This directly yields the following results.

Corollary 2.5.7. *Suppose that $f \in \mathscr{C}^\infty(\mathrm{T}^*Q)$ is a Hamiltonian constant of motion, such as a Hamiltonian constraint. Then, $(K \cdot f)$ is a Lagrangian constraint.*

A complete classification of Lagrangian and Hamiltonian constraints appearing in the constraint algorithms for singular dynamical systems can also achieved using the K operator. In fact, *all* the Lagrangian constraints can be obtained from the Hamiltonian ones using the time-evolution operator [6].

3. Contact mechanics

This chapter is devoted to introducing the basic concepts of contact geometry and contact Hamiltonian systems. We define contact manifolds and Reeb vector fields, state the existence and uniqueness theorem for the Reeb vector field, and demonstrate the equivalence of several formulations of the contact Hamiltonian equations—two of which do not involve the Reeb vector field. With these structures, we provide a geometric framework suitable for studying mechanical systems with dissipation. In particular, we present the dissipative Lagrangian formalism derived from the Herglotz variational principle [39, 57, 58], formulated in terms of (pre)contact manifolds. As in the classical case, under suitable regularity conditions, we construct a corresponding Hamiltonian formalism, now defined on an extended cotangent bundle.

The main references used for this chapter are [25, 34, 62].

3.1 Contact geometry

Definition 3.1.1. Let M be a manifold of dimension $2n+1$, and $\eta \in \Omega^1(M)$ a differential 1-form such that

$$\eta \wedge (\mathrm{d}\eta)^n \neq 0\,,$$

i.e. $\eta \wedge (\mathrm{d}\eta)^n$ is a volume form. Then, we say that η is a *contact form* and that the pair (M, η) is a *contact manifold*.

Remark 3.1.2. In the literature [4], some authors use a different definition of contact manifolds. Namely, they define a contact manifold as a pair $(M, \mathcal{H})$, consisting of an odd-dimensional manifold M and a *contact distribution* $\mathcal{H}$. A contact distribution is a rank $2n$ distribution on a $(2n{+}1)$-dimensional manifold. We do not use this definition because most of the results are either of local character, and then we can find a 1-form η such that it locally defines the contact distribution with its kernel; or because in the contact Lagrangian and Hamiltonian formalisms the contact distribution is directly spanned from a suitable 1-form. Our definition of contact manifold sometimes receives the name of *co-oriented contact manifold* [45, 46].

Given a contact manifold (M, η), the contact form η defines a vector bundle isomorphism

$$\begin{aligned} B\colon \mathrm{T}M &\longrightarrow \mathrm{T}^*M\,,\\ v &\longmapsto i_v\mathrm{d}\eta + (i_v\eta)\eta\,, \end{aligned}$$

which induces the $\mathscr{C}^\infty(M)$-module isomorphism $B\colon \mathfrak{X}(M) \to \Omega^1(M)$. This isomorphism provides a decomposition of $\mathrm{T}M$ as

$$\mathrm{T}M = \operatorname{Ker}\eta \oplus \operatorname{Ker}\mathrm{d}\eta\,.$$

Theorem 3.1.3 (Contact Darboux theorem)**.** *Let (M, η) be a contact manifold. Then, for each point $x \in M$, there exists a chart $(U_x, (q^i, p_i, s))$, with $1 \le i \le n$, in which the contact form reads*

$$\eta = \mathrm{d}s - p_i\mathrm{d}q^i\,.$$

Such a chart is called a contact chart and the coordinates it defines are the so-called Darboux or canonical coordinates of the contact manifold (M, η).

Definition 3.1.4 (Reeb vector field)**.** The Reeb vector field $R \in \mathfrak{X}(M)$ of a contact manifold (M, η) is the only vector field such that $\flat(R) = \eta$ or, equivalently,

$$\begin{cases} i_R \eta = 1\,, \\ i_R \mathrm{d}\eta = 0\,. \end{cases}$$

Note that the unicity comes from $\flat$ being an isomorphism, and that, in Darboux coordinates, the Reeb vector field is written as

$$R = \frac{\partial}{\partial s}\,.$$

Now, we give an example of a family of important contact manifolds, which are obtained from symplectic manifolds.

Example 3.1.5. Recall that a symplectic manifold is a pair (N, ω), with N a manifold of even dimension and $\omega \in \Omega^2(N)$ a closed and non-degenerate 2-form. As a consequence of ω being closed (by the Poincaré lemma) we can, at least locally, write $\omega = -\mathrm{d}\theta$, with θ a certain 1-form.

One can easily *contactify* a symplectic manifold (N, ω), just by considering the manifold $M = N \times \mathbb{R}$ and the 1-form $\eta \in \Omega^1(M)$, given by $\eta = \mathrm{d}s - \theta$, where s denotes the Cartesian coordinate of $\mathbb{R}$ and θ the pull-back of $\theta \in \Omega^1(N)$ to M. Then, the pair (M, η) is a contact manifold, called the *contactization* of (N, ω).

An important particular case of this consists in considering $(\mathrm{T}^*Q, \omega_Q)$, with $\omega_Q = -\mathrm{d}\theta_Q$ and θ_Q the canonical 1-form on T^*Q. The contactization of this symplectic manifold gives the contact manifold $(\mathrm{T}^*Q \times \mathbb{R}, \eta)$. With canonical coordinates (q^i, p_i) on T^*Q, we have $\eta = ds - p_i \mathrm{d}q^i$; and therefore $\mathrm{d}\eta = \mathrm{d}q^i \wedge \mathrm{d}p_i$ and $R = \frac{\partial}{\partial s}$.

Definition 3.1.6. A *contact Hamiltonian system* is a triple (M, η, H), with (M, η) a contact manifold and $H \in \mathscr{C}^\infty(M)$, the so-called *Hamiltonian function*.

Theorem 3.1.7 (Contact Hamiltonian equations)**.** *Given a contact Hamiltonian system (M, η, H), there exists a unique vector field $X_H \in \mathfrak{X}(M)$, called the contact Hamiltonian vector field, such that*

$$\begin{cases} i_{X_H} \mathrm{d}\eta = \mathrm{d}H - (\mathscr{L}_R H)\eta\,, \\ i_{X_H} \eta = -H\,. \end{cases} \tag{3.1}$$

The integral curves of the contact Hamiltonian vector field, $\xi \colon I \subset \mathbb{R} \to M$, are the solutions to the equations

$$\begin{cases} i_{\dot{\xi}} \mathrm{d}\eta = (\mathrm{d}H - (\mathscr{L}_R H)\eta) \circ \xi\,, \\ i_{\dot{\xi}} \eta = -H \circ \xi\,, \end{cases} \tag{3.2}$$

with $\dot{\xi}$ being the canonical lift of the path ξ to $\mathrm{T}M$. Equations (3.1) and (3.2) are the contact Hamilton equations.

If we write the last theorem in canonical Darboux coordinates, we have that the contact Hamiltonian vector field can be expressed, locally, as

$$X_H = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \left(\frac{\partial H}{\partial q^i} + p_i \frac{\partial H}{\partial s}\right)\frac{\partial}{\partial p_i} + \left(p_i \frac{\partial H}{\partial p_i} - H\right)\frac{\partial}{\partial s}\,,$$

and so, a path $\xi\colon I\subset\mathbb{R}\to M, \xi(t)=(q^i(t),p_i(t),s(t))$ is a an integral curve of X_H if and only if

$$\begin{aligned}
\frac{\mathrm{d}q^i(t)}{\mathrm{d}t} &= \frac{\partial H}{\partial p_i}(\xi(t))\,,\\
\frac{\mathrm{d}p_i(t)}{\mathrm{d}t} &= -\left(\frac{\partial H}{\partial q^i}+p_i\frac{\partial H}{\partial s}\right)\circ(\xi(t))\,,\\
\frac{\mathrm{d}s(t)}{\mathrm{d}t} &= \left(p_i\frac{\partial H}{\partial p_i}-H\right)\circ(\xi(t))\,,
\end{aligned}$$

that is, it satisfies the contact Hamiltonian equations.

In the symplectic case, remember that the Hamiltonian was a conserved quantity of the system, i.e. it was conserved along integral curves of the Hamiltonian vector field. This is not the case in contact Hamiltonian systems, as one can check easily that

$$\mathscr{L}_{X_H}H=-(\mathscr{L}_R H)H\,.$$

Proposition 3.1.8. *Let (M,η,H) be a contact Hamiltonian system. Then, for $X_H\in\mathfrak{X}(M)$, a vector field on M, the following statements are equivalent:*

1. *The vector field satisfies*

$$\begin{cases}(i_X\mathrm{d}\eta)\wedge\eta=(\mathrm{d}H)\wedge\eta\,,\\ i_X\eta=-H\,.\end{cases}\tag{3.3}$$

2. *The vector field satisfies*

$$\begin{cases}i_X(\eta\wedge\mathrm{d}\eta)=\Omega\,,\\ i_X\eta=-H\,.\end{cases}\tag{3.4}$$

   where Ω is the 2-form on M defined as $\Omega:=-H\mathrm{d}\eta+\mathrm{d}H\wedge\eta$.

3. *The vector field satisfies*

$$\begin{cases}i_X\mathrm{d}\eta=\mathrm{d}H-(\mathscr{L}_R H)\eta\,,\\ i_X\eta=-H\,,\end{cases}$$

   where R is the so-called Reeb vector field, the unique vector field satisfying

$$i_R\eta=1\,,\qquad i_R\mathrm{d}\eta=0\,.$$

4. *The vector field satisfies*

$$B(X)=\mathrm{d}H-(\mathscr{L}_R H+H)\eta\,.\tag{3.5}$$

5. *The vector field satisfies*

$$\begin{cases}\mathscr{L}_X\eta=-(\mathscr{L}_R H)\eta\,,\\ i_X\eta=-H\,.\end{cases}\tag{3.6}$$

Proof. To see that 1. implies 2. we just need to see that

$$i_X(\eta\wedge\mathrm{d}\eta)=(i_X\eta)\wedge\mathrm{d}\eta+(i_X\mathrm{d}\eta)\wedge\eta=-H\mathrm{d}\eta+\mathrm{d}H\wedge\eta=\Omega\,,$$

where we have used Equations (3.3) in the third equality.

In order to prove that 2. implies 3. we just need to contract the first equation of (3.4) by the Reeb vector field. Indeed, it is easy to see that $i_R(\eta \wedge \mathrm{d}\eta) = \mathrm{d}\eta$, and using this we obtain

$$i_R i_X(\eta \wedge \mathrm{d}\eta) = -i_X \mathrm{d}\eta$$

and on the other side of the equation we have

$$i_R \Omega = (\mathscr{L}_R H)\eta - \mathrm{d}H\,.$$

Thus, equating both sides we get

$$i_X \mathrm{d}\eta = \mathrm{d}H - (\mathscr{L}_R H)\eta\,,$$

which is exactly what we wanted.

To prove that 3. implies 4., it is just a matter of using the relations we have. Indeed,

$$B(X) = i_X \mathrm{d}\eta + (i_X\eta)\eta = \mathrm{d}H - (\mathscr{L}_R H)\eta - H\eta\,.$$

To prove that 4 implies 5, we first contract by the Reeb vector field. This yields

$$i_X\eta = i_R B(X) = -H\,,$$

which is the second equation. Now, substituting into (3.5), we get

$$i_X \mathrm{d}\eta = \mathrm{d}H - (\mathscr{L}_R H)\eta$$

and so,

$$\mathscr{L}_X\eta = i_X \mathrm{d}\eta + \mathrm{d}i_X\eta = i_X\mathrm{d}\eta - \mathrm{d}H\,.$$

The only thing left to prove is that 5. implies 1. Note that, if we have Equations (3.6) then

$$-(\mathscr{L}_R H)\eta = \mathscr{L}_X\eta = i_X \mathrm{d}\eta + \mathrm{d}i_X\eta = i_X\mathrm{d}\eta - \mathrm{d}H\,,$$

which implies that

$$i_X \mathrm{d}\eta = \mathrm{d}H - (\mathscr{L}_R H)\,,$$

and so

$$(i_{X_H}\mathrm{d}\eta)\wedge\eta = (\mathrm{d}H)\wedge\eta\,,$$

thus proving all the equivalences. □

The importance of Equations (3.3) and (3.4) will be made more apparent in Chapter 4, when we deal with *precontact systems*. Precontact manifolds do not necessarily define a Reeb vector field, and when they do it is not unique. Hence, the possibility of writing the contact Hamiltonian equations without the dependence of the Reeb vector field proves very useful in the singular setting.

There exists another equivalent way to formulate the contact Hamilton equations, which also does not use the Reeb vector field. However, it does not work necessarily at every point, just when the Hamiltonian function H does not vanish.

Proposition 3.1.9. *Let (M, η, H) be a contact Hamiltonian system. Consider the open set given by $U = \{p \in M \mid H(p) \neq 0\}$ and the 2-form $\Omega \in \Omega^2(M)$ defined as $\Omega = -H\mathrm{d}\eta + \mathrm{d}H \wedge \eta$ on U. A vector field $X \in \mathfrak{X}(U)$ is the contact Hamiltonian vector field if and only if, it verifies*

$$\begin{cases} i_X\Omega = 0\,, \\ i_X\eta = -H\,. \end{cases} \tag{3.7}$$

Note that, if we consider the case where $H(p) = 0$, then both second equations in (3.1) and in (3.7) yield $i_{X_p}\eta = -H(p) = 0$. But $i_{X_p}\mathrm{d}\eta = \mathrm{d}H(p) - (\mathscr{L}_{R_p}H)\eta_p$, from (3.1), and $i_X\Omega = (\mathscr{L}_{X_p}H)\eta_p = 0$, from (3.7), are not equivalent in this case. The first implies the second one, but not in the contrary.

3.2 Contact Lagrangian formalism

In this section, we develop a contact Lagrangian formalism, in order to deal with mechanical systems with dissipation. The construction is similar to the one established (in Section 2.2) for classical mechanics via symplectic manifolds.

Throughout this section, we consider Q, an n-dimensional manifold, and its respective tangent bundle $\mathrm{T}Q$. Our objective is to equip the product manifold $\mathrm{T}Q\times\mathbb{R}$, which we assume has local coordinates (q^i, v^i, s), with a contact structure associated to a given *contact Lagrangian* $L \in \mathscr{C}^\infty(\mathrm{T}Q\times\mathbb{R})$. We will be able to do this if the contact Lagrangian function satisfies a regularity condition. Otherwise, we are left with a *precontact structure*, which is studied in the following chapter.

Through the so-called Herglotz's principle, one can derive Herglotz's equations, sometimes also referred to as the *generalized Euler–Lagrange or contact Euler–Lagrange* equations. In local coordinates, a path $\gamma\colon I \to Q\times\mathbb{R}$, $\gamma(t) = (q(t), s(t))$, satisfies the generalized Euler–Lagrange equations if

$$
\begin{gathered}
\dot{s}(t) = L\circ\widetilde{\gamma}(t)\,,\\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial v^i}\circ\widetilde{\gamma}(t)\right) - \frac{\partial L}{\partial q^i}\circ\widetilde{\gamma}(t) = \frac{\partial L}{\partial s}\frac{\partial L}{\partial v^i}\circ\widetilde{\gamma}(t)\,,
\end{gathered}
$$

where $\widetilde{\gamma}\colon I \to \mathrm{T}Q\times\mathbb{R}$ is the so-called *prolongation* of γ to $\mathrm{T}Q\times\mathbb{R}$. That is, $\widetilde{\gamma} = \rho\circ\dot{\gamma}$, where $\rho\colon \mathrm{T}(\mathrm{T}Q\times\mathbb{R}) \to \mathrm{T}Q\times\mathbb{R}$ and $\dot{\gamma}$ is the canonical lift of the path. When a path on $\mathrm{T}Q\times\mathbb{R}$ can be expressed as the prolongation of some path defined on $Q\times\mathbb{R}$, it is said to be *holonomic*, in local coordinates this is equivalent to $\widetilde{\gamma}(t) = (q(t), \dot{q}(t), s(t))$.

Note that these equations do indeed generalize the usual Euler–Lagrange equations, as if the Lagrangian function does not depend on the variable s then we retrieve the Euler–Lagrange equations.

Similarly to the what we did in the last chapter, we aim to express the generalized Euler–Lagrange equations through the contact Hamiltonian equations (3.1). This formalism will allow us to express the generalized Euler–Lagrange equations in a geometric, coordinate-free manner.

To do so, first note that we can write $\mathrm{T}(\mathrm{T}Q\times\mathbb{R}) = (\mathrm{T}(\mathrm{T}Q)\times\mathbb{R})\otimes(\mathrm{T}Q\times\mathrm{T}\mathbb{R})$, and so every operation acting on tangent vectors on $\mathrm{T}Q$ can act on tangent vectors on $\mathrm{T}Q\times\mathbb{R}$. Therefore, we have a natural extension of the geometric structures defined on $\mathrm{T}Q$ to $\mathrm{T}Q\times\mathbb{R}$; in particular, we have the induced *vertical endomorphism* $J\colon \mathrm{T}(\mathrm{T}Q\times\mathbb{R}) \to \mathrm{T}(\mathrm{T}Q\times\mathbb{R})$ and the *Liouville vector field* $\Delta \in \mathfrak{X}(\mathrm{T}Q\times\mathbb{R})$, which are expressed in local coordinates as

$$
J = \frac{\partial}{\partial v^i}\otimes\mathrm{d}q^i\,,\qquad \Delta = v^i\frac{\partial}{\partial v^i}\,.
$$

With these, we can define *second-order vector fields* $X \in \mathfrak{X}(\mathrm{T}Q\times\mathbb{R})$ as the vector fields satisfying $J\circ X = \Delta$. Their local expression is given by

$$
X = v^i\frac{\partial}{\partial q^i} + f^i\frac{\partial}{\partial v^i} + g\frac{\partial}{\partial s}.
$$

Definition 3.2.1. The so-called *Lagrangian 1-form* η_L is defined by

$$\eta_L = \mathrm{d}s - \theta_L \,.$$

Here, θ_L is the Cartan 1-form associated with L, which is given by $\theta_L = {}^tJ \circ \mathrm{d}L \in \Omega^1(\mathrm{T}Q \times \mathbb{R})$. With this, the *Lagrangian 2-form* is defined simply as $\mathrm{d}\eta_L$.

Remark 3.2.2. Note that, the Cartan 2-form $\omega_L = -\mathrm{d}\theta_L$ coincides with the differential of the Lagrangian 1-form, i.e. $\mathrm{d}\eta_L = \omega_L$.

In natural coordinates, we can write the contact Lagrangian forms as

$$\eta_L = \mathrm{d}s - \frac{\partial L}{\partial v^i}\mathrm{d}q^i \,,$$

and,

$$\mathrm{d}\eta_L = -\frac{\partial^2 L}{\partial s \partial v^i}\mathrm{d}s \wedge \mathrm{d}q^i - \frac{\partial^2 L}{\partial q^j \partial q^i}\mathrm{d}q^j \wedge \mathrm{d}q^i - \frac{\partial^2 L}{\partial v^j \partial v^i}\mathrm{d}v^j \wedge \mathrm{d}q^i \,.$$

Definition 3.2.3 (Contact Lagrangian system)**.** Given a manifold Q and a *contact Lagrangian function* $L\colon \mathrm{T}Q \times \mathbb{R} \to \mathbb{R}$, we say that $(\mathrm{T}Q \times \mathbb{R}, \eta_L, E_L)$ is a *Lagrangian system*. The function $E_L = \Delta(L) - L$ is the *contact Lagrangian energy*, with $\Delta \in \mathfrak{X}(\mathrm{T}Q \times \mathbb{R})$ the Liouville vector field.

Remark 3.2.4. Note that, to construct the contact Lagrangian system, we *contactified* (as shown in Example 3.1.5) the symplectic system constructed in Section 2.2.

Similar to the standard case, the 1-form η_L is *not always a contact form*. We need to add an additional regularity condition on the contact Lagrangian function to assure that this is the case.

Definition 3.2.5. The *Legendre transformation* of a contact Lagrangian function $L\colon \mathrm{T}Q \times \mathbb{R} \to \mathbb{R}$ is its fibre derivative, considered as a function on the vector bundle $\tau_0\colon \mathrm{T}Q \times \mathbb{R} \to Q \times \mathbb{R}$. That is, $\mathcal{F}L\colon \mathrm{T}Q \times \mathbb{R} \to \mathrm{T}^*Q \times \mathbb{R}$ is given by

$$\mathcal{F}L(v_q, s) = (\mathcal{F}L(\cdot, s)(v_q), s) \,.$$

In natural coordinates

$$\mathcal{F}L(q^i, v^i) = \left(q^i, \frac{\partial L}{\partial v^i}, s\right) \,.$$

Definition 3.2.6. A contact Lagrangian function $L\colon \mathrm{T}Q \times \mathbb{R} \to \mathbb{R}$ is *regular* if the following equivalent conditions hold:

1. The Legendre transformation $\mathcal{F}L\colon \mathrm{T}Q \times \mathbb{R} \to \mathrm{T}^*Q \times \mathbb{R}$ is a local diffeomorphism.
2. At each $u_q \in \mathrm{T}Q$ and $s \in \mathbb{R}$ the fibre Hessian $\mathcal{F}^2L(u_q, s)$ is a non-degenerate bilinear form.
3. In natural coordinates the Hessian matrix $\left(\dfrac{\partial^2 L}{\partial v^i \partial v^j}\right)$ is everywhere non-singular.
4. The associated Lagrangian 1-form η_L is a contact form.

We say that the Lagrangian system $(\mathrm{T}Q \times \mathbb{R}, \eta_L, E_L)$ is *regular* if it is a contact Hamiltonian system, that is, if the contact Lagrangian function L is regular. If the contact Lagrangian is regular, then the Reeb vector field is uniquely defined by the conditions

$$\begin{cases} i_{R_L} \mathrm{d}\eta_L = 0\,, \\ i_{R_L} \eta_L = 1\,. \end{cases}$$

Its has local expression

$$R_L = \frac{\partial}{\partial s} - W^{ji} \frac{\partial^2 L}{\partial s \partial v^j} \frac{\partial}{\partial v^i}\,,$$

where (W^{ij}) is the inverse matrix of the Hessian matrix $\left(\frac{\partial^2 L}{\partial v^i \partial v^j}\right)$.

Therefore, for a regular contact Lagrangian, there exists a unique *contact Lagrangian vector field* such that

$$\begin{cases} i_{X_L} \mathrm{d}\eta_L = \mathrm{d}E_L - (\mathscr{L}_{R_L} E_L)\eta_L\,, \\ i_{X_L} \eta_L = -E_L\,, \end{cases} \tag{3.8}$$

these are the so-called *contact Euler–Lagrange equations*.

Proposition 3.2.7. *If the contact Lagrangian L is regular, then X_L is a second-order differential vector field and its integral curves satisfy the generalized Euler–Lagrange equations.*

In local coordinates, this means that $X_L = v^i \frac{\partial}{\partial q^i} + f^i \frac{\partial}{\partial v^i} + g \frac{\partial}{\partial s}$ with

$$g = L\,,$$
$$\frac{\partial^2 L}{\partial v^j \partial v^i} f^j + \frac{\partial^2 L}{\partial q^j \partial v^i} v^j + \frac{\partial^2 L}{\partial s \partial v^i} L - \frac{\partial L}{\partial q^i} = \frac{\partial L}{\partial s} \frac{\partial L}{\partial v^i}$$

Proof. The proof of this proposition follows directly from writing the coordinate expressions. See [34] for more details. ☐

Note that, the generalized Euler–Lagrange equations, as they come from a variational principle, exist without dependence on the regularity of the Lagrangian. As a matter of fact, we can express such equations in a geometric way without the need of a regular Lagrangian. In general, if the Lagrangian is singular, the solutions to the generalized Euler–Lagrange equations can be seen as integral curves of second-order vector fields satisfying

$$\begin{cases} i_{X_L} \mathrm{d}\eta_L = \mathrm{d}E_L + \left(\dfrac{\partial L}{\partial s}\right) \eta_L\,, \\ i_{X_L} \eta_L = -E_L\,. \end{cases}$$

These are well-defined, as $\frac{\partial L}{\partial s}$ is defined canonically in $\mathrm{T}Q \times \mathbb{R}$, and they are equivalent to (3.8), as one can check that in the regular case $\mathscr{L}_{R_L} E_L = -\frac{\partial L}{\partial s}$. If the Lagrangian is singular, these conditions do not imply that the vector field is second-order, and thus the condition must be added. Also, in the singular case, there is no existence of solutions necessarily at every point, and where there is the uniqueness is not guaranteed either. This demands a more careful study of both the Lagrangian and Hamiltonian formalisms in the non-regular cases, which is further discussed in the next chapter of this thesis.

3.3 Contact Hamiltonian formalism

Let us analyse how to construct a contact Hamiltonian formalism that is related to the contact Lagrangian formalism we exposed in the last section. That is, we aim to provide a contact Hamiltonian system defined on $\mathrm{T}^*Q\times\mathbb{R}$ which is in correspondence with the Lagrangian system $(\mathrm{T}Q\times\mathbb{R}, \eta_L, E_L)$.

Recall, from Example 3.1.5, that defined on $\mathrm{T}^*Q\times\mathbb{R}$ we have a canonical contact form

$$\eta_Q = \mathrm{d}s - \theta_Q \,,$$

where $\theta_Q \in \Omega^1(\mathrm{T}^*Q\times\mathbb{R})$ is the pull-back of the canonical 1-form on the cotangent bundle to $\mathrm{T}^*Q\times\mathbb{R}$.

A direct computation shows that

$$\eta_L = \mathcal{F}L^*(\eta_Q) \,,$$

and, if the contact Lagrangian is regular, then energy the energy E_L is, at least locally, $\mathcal{F}L$-projectable. That is, we can always find, locally, a *contact Hamiltonian function* $H\colon \mathrm{T}^*Q\times\mathbb{R}\to\mathbb{R}$ such that

$$E_L = \mathcal{F}L^*(H) \,.$$

We say that a contact Lagrangian function L is hyperregular if $\mathcal{F}L$ is a *global* diffeomorphism. In this case, the energy is globally $\mathcal{F}L$-projectable and the contact Hamiltonian system $(\mathrm{T}^*Q\times\mathbb{R}, \eta_Q, H)$ satisfies

$$\mathcal{F}L_*(R_L) = R \,,$$

and also

$$\mathcal{F}L_*(X_L) = X_H \,,$$

for $X_H \in \mathfrak{X}(\mathrm{T}^*Q\times\mathbb{R})$ the contact Hamiltonian vector field associated to the system. This, in other words, means that the two formalisms are $\mathcal{F}L$-related, and the map $\xi \mapsto \mathcal{F}L\circ\widetilde{\xi}$, where $\xi\colon I\to Q\times\mathbb{R}$ is a solution to the contact Euler–Lagrange equations, sends solutions to solutions. Conversely, if $\psi\colon I\to \mathrm{T}^*Q\times\mathbb{R}$ is a integral path of X_H, then the map $\psi\mapsto\pi_0\circ\psi$, with $\pi_0\colon \mathrm{T}^*Q\times\mathbb{R}\to Q\times\mathbb{R}$ the canonical projection, also sends solutions of the contact Hamiltonian formalism to solutions of the Euler–Lagrange equations.

We study the non-regular case in the next chapter of this thesis. Let us now point out that, regardless of the regularity of the Lagrangian, any solution on $\mathrm{T}^*Q\times\mathbb{R}$ which is $\mathcal{F}L$-related to a solution of the contact Euler–Lagrange equations has to necessarily satisfy the equations

$$\begin{cases} p = \dfrac{\partial L}{\partial v}\,, \\ \dot p = \dfrac{\partial L}{\partial s}\dfrac{\partial L}{\partial v} + \dfrac{\partial L}{\partial q}\,, \\ \dot s = L\,, \end{cases}$$

we call these equations the *Herglotz–Dirac* equations.

4. Precontact systems

In this chapter, we study precontact systems and their associated geometric structures. We provide a definition of a precontact manifold that agrees with the one presented in [46] and generalizes those considered in [26, 61] and [22]. In the first section, we define and present results related to the *class* and the *rank* of a differential form, as presented in [40]. In the rest of the chapter, we use these preliminary results to characterize precontact manifolds and provide information about the existence of Reeb vector fields associated with the precontact structure. Last, using the new contact Hamiltonian equations (3.3), we formulate well-defined dynamics for precontact systems. In addition, we examine the existence of functions that allow for a change in the parity of the class.

4.1 Class of a differential form

Definition 4.1.1. Let $\alpha \in \Omega^p(M)$ be a p-form. The *kernel* of α at a point x is

$$\operatorname{Ker}\alpha(x) = \{v \in \mathrm{T}_x M \mid i_v\alpha(x) = 0\}\,.$$

We can define its annihilator as

$$(\operatorname{Ker}\alpha(x))^\circ = \{u \in \mathrm{T}^*_x M \mid i_v u = 0,\ \forall\, v \in \operatorname{Ker}\alpha(x)\}\,.$$

The dimension of the annihilator, also called the *rank* of α, is the codimension of the kernel.

Note that any nowhere-vanishing 1-form $\eta \in \Omega^1(M)$ defines a codimension 1 distribution at every point by

$$\operatorname{Ker}\eta(x) = \{v \in \mathrm{T}_x M \mid i_v\eta(x) = 0\}\,,$$

so the *rank of any nowhere-vanishing 1-form is always equal to 1*. Actually, for any nowhere-vanishing function $f \in \mathscr{C}^\infty(M)$, the 1-form $f\eta \in \Omega^1(M)$ defines the same distribution as η.

Similarly, given a codimension 1 distribution $D \subset \mathrm{T}M$, one can find (at least locally) a 1-form η such that its kernel, and consequently that of all its *conformal* 1-forms $f\eta$, is precisely the distribution $D \subset \mathrm{T}M$.

Theorem 4.1.2. *Given a p-form $\alpha \in \Omega^p(M)$, on a manifold M, the image of the multilinear map*

$$\begin{aligned} h\colon \mathrm{T}_x M \otimes \overset{(p-1)}{\cdots} \otimes \mathrm{T}_x M &\longrightarrow \mathrm{T}^*_x M \\ (X_1, \dots, X_{p-1}) &\longmapsto i_{X_1} \dots i_{X_{p-1}}\alpha \end{aligned}$$

generates the whole annihilator $(\operatorname{Ker}\alpha(x))^\circ$.

In particular, last theorem implies that for a 1-form $\alpha \in \Omega^1(M)$ we have that

$$(\operatorname{Ker}\alpha(x))^\circ = \langle \alpha(x) \rangle\,.$$

On the other hand, if $\omega \in \Omega^2(M)$ is a 2-form, then

$$(\operatorname{Ker}\omega(x))^\circ = \langle i_{e_1}\omega(x), \dots, i_{e_m}\omega(x) \rangle\,,$$

where m is the dimension of M and $e_1, \dots e_m \in \mathrm{T}_x M$ is a basis of vectors of $\mathrm{T}_x M$.

Theorem 4.1.3. *Let $\omega \in \Omega^2(M)$ be 2-form on M, then the dimension of $(\operatorname{Ker}\omega(x))^\circ$ is always even. That is, the rank of ω is even at every point of M.*

Also, the rank of the 2-form is $2r$ if and only if it satisfies

$$(\omega)^r \neq 0\,, \qquad (\omega)^{r+1} = 0\,.$$

Definition 4.1.4. Let $\eta \in \Omega^p(M)$ be a p-form on M. Its *characteristic distribution* at a point $x \in M$ is given by

$$\mathcal{C}(x) = \operatorname{Ker}\eta(x) \cap \operatorname{Ker}\mathrm{d}\eta(x)\,,$$

where $\mathrm{d}\eta \in \Omega^{p+1}(M)$ is the exterior derivative of η.

The annihilator of the characteristic distribution $\mathcal{C}^\circ(x)$ is, then, given by

$$\mathcal{C}^\circ(x) = (\operatorname{Ker}\eta(x) \cap \operatorname{Ker}\mathrm{d}\eta(x))^\circ = (\operatorname{Ker}\eta(x))^\circ + (\operatorname{Ker}\mathrm{d}\eta(x))^\circ\,.$$

We say that a p-form is *of class r* at a point $x \in M$ if the codimension of its characteristic distribution is equal to r at $x \in M$. The codimension of the characteristic distribution is equal to the dimension of its annihilator.

Remark 4.1.5. Note that, if $\omega \in \Omega^p(M)$ is a *closed* p-form on M, then

$$\mathcal{C} = \operatorname{Ker}\omega \cap \operatorname{Ker}\mathrm{d}\omega = \operatorname{Ker}\omega\,,$$

because $\mathrm{d}\omega = 0$. This, in particular, implies that if ω is a closed p-form, then *the rank of ω is equal to its class.*

As for any p-form $\eta \in \Omega^p(M)$, its exterior derivative $\mathrm{d}\eta \in \Omega^{p+1}(M)$ is obviously closed, $\mathrm{d}\eta$ has class equal to its rank. In particular, by Theorem 4.1.3, this implies that if $\eta \in \Omega^1(M)$ is a 1-form, its exterior derivative $\mathrm{d}\eta \in \Omega^2(M)$ necessarily has even class.

4.2 Precontact structures

Definition 4.2.1 (Precontact manifold)**.** Let M be a smooth manifold and $\eta \in \Omega^1(M)$ a nowhere-vanishing 1-form on M. Then, we say that the pair (M, η) is a *precontact manifold of rank $2r+1$* if it satisfies

$$\eta \wedge (\mathrm{d}\eta)^r \neq 0, \qquad \eta \wedge (\mathrm{d}\eta)^{r+1} = 0\,,$$

at every point of M.

In general, a 1-form defined on a manifold can have different precontact rank at different points. In our definition we consider only the case where the precontact manifold has constant rank at every point of the manifold.

Remark 4.2.2. With this notation, if M has odd dimension $\dim M = 2n+1$, then a precontact manifold (M, η) of rank $2n+1$ is a contact manifold.

The characterization of contact manifolds by

$$\eta \wedge (\mathrm{d}\eta)^n \neq 0\,,$$

is sometimes referred to as *maximal non-integrability* of the 1-form. This is because of the Frobenius theorem for differential forms, which when applied to a single one-form η establishes that

$$P(x) = \{v_x \in \mathrm{T}_x M \mid i_{v_x}\eta = 0\}\,,$$

is integrable if and only if $\eta \wedge \mathrm{d}\eta = 0$. Hence, the distribution defined by the kernel of η is integrable if and only if the rank of the precontact manifold is 1. In this sense, the rank of a precontact manifold indicates how "far" the system is from being integrable.

Let us now explore an equivalent characterization of precontact manifolds, in terms of the class of the 1-form. This will provide more insight into some of their geometric properties. From now on, we only consider differential forms such that they have *constant class at every point*.

Note that, in the regular contact case, the kernels of both the 1-form η and of the 2-form $\mathrm{d}\eta \in \Omega^2(M)$ play a crucial role. Indeed, we have that the dimension of $\operatorname{Ker} \mathrm{d}\eta$ is equal to 1 and that

$$\mathrm{T}M = \operatorname{Ker} \eta \oplus \operatorname{Ker} \mathrm{d}\eta \,.$$

This implies that

$$\mathcal{C} = \operatorname{Ker} \eta \cap \operatorname{Ker} \mathrm{d}\eta = \emptyset$$

in the contact case, and so its annihilator $\mathcal{C}^\circ$ has maximal dimension $2n+1$, and therefore the class of η is $2n+1$. This result is strongly related to the existence and uniqueness of the Reeb vector field.

We can use the results and definitions presented in the last section to further study the class of a general precontact manifold. Let $\eta \in \Omega^1(M)$ be a 1-form, recall that its characteristic distribution is defined as

$$\mathcal{C} = \operatorname{Ker} \eta \cap \operatorname{Ker} \mathrm{d}\eta$$

and its annihilator is given by

$$\mathcal{C}^\circ = (\operatorname{Ker} \eta)^\circ + (\operatorname{Ker} \mathrm{d}\eta)^\circ \,.$$

To study the dimension of $\mathcal{C}^\circ$, i.e. the class of η, it is important to note that:

- The rank of a 2-form is necessarily even, by Theorem 4.1.3.
- The rank of a nowhere-vanishing 1-form is equal to one. In particular, the class of η is always bigger or equal than one.

These remarks directly imply that *if the class of the 1-form η is odd*, then

$$(\operatorname{Ker} \eta)^\circ \not\subset (\operatorname{Ker} \mathrm{d}\eta)^\circ \,,$$

which means that *there cannot exist* any $X \in \operatorname{Ker} \eta$ satisfying

$$i_X \mathrm{d}\eta = \eta \,,$$

by Theorem 4.1.2. One can also invert the inclusion, by taking the annihilator on both sides, to obtain that

$$\operatorname{Ker} \eta \not\supset \operatorname{Ker} \mathrm{d}\eta \,,$$

which implies the existence of Reeb vector fields in this case, that is, vector fields satisfying

$$i_R \eta = 1 \,, \qquad i_R \mathrm{d}\eta = 0 \,.$$

On the other hand, *if the class of η is even*, then necessarily we have

$$(\operatorname{Ker} \eta)^\circ \subset (\operatorname{Ker} \mathrm{d}\eta)^\circ \,,$$

which immediately implies, that there exist $\Delta \in \mathfrak{X}(M)$, such that

$$i_\Delta \mathrm{d}\eta = \eta \,, \tag{4.1}$$

by Theorem 4.1.2. If a vector field satisfies Property (4.1), then we say that it is a *Liouville or a Liouville-type vector field* for η. By inverting the inclusion, one obtains

$$\operatorname{Ker}\eta \supset \operatorname{Ker}\mathrm{d}\eta\,,$$

in this case. This directly implies that *there cannot exist any Reeb vector fields* if the class of η is even.

The next propositions relate the class of a 1-form and the rank of the precontact manifold it defines.

Proposition 4.2.3. *Let $\eta \in \Omega^1(M)$ be a 1-form on a manifold M. Then, η has class $2r$ if and only if it satisfies*

$$\begin{cases} \eta \wedge (\mathrm{d}\eta)^r = 0\,, \\ (\mathrm{d}\eta)^r \neq 0\,. \end{cases} \tag{4.2}$$

Proof. Assume that η has class $2r$. This directly implies that the rank of $\mathrm{d}\eta$ is $2r$, and so

$$(d\eta)^r \neq 0\,, \qquad (\mathrm{d}\eta)^{r+1} = 0\,,$$

by Theorem 4.1.3.

It only remains to show that $\eta \wedge (\mathrm{d}\eta)^r = 0$. As the class is even, we have that

$$(\operatorname{Ker}\eta)^\circ \subset (\operatorname{Ker}\mathrm{d}\eta)^\circ\,,$$

and thus, the existence of a Liouville vector field $\Delta \in \mathfrak{X}(M)$ satisfying

$$i_\Delta \mathrm{d}\eta = \eta\,.$$

If we assume that $\eta \wedge (\mathrm{d}\eta)^r \neq 0$, we reach a contradiction by contracting with Δ. Indeed,

$$0 = i_\Delta (\mathrm{d}\eta)^{r+1} = (r+1)\eta \wedge (\mathrm{d}\eta)^r\,,$$

and therefore, necessarily, $\eta \wedge (\mathrm{d}\eta)^r = 0$.

For the converse, first note that the assumptions (4.2) are enough to verify that $\mathrm{d}\eta$ has rank $2r$, because

$$(\mathrm{d}\eta)^r \neq 0\,, \qquad \mathrm{d}(\eta \wedge (\mathrm{d}\eta)^r) = (\mathrm{d}\eta)^{r+1} = 0\,.$$

This already implies that the class of η is either $2r$ or $2r+1$. Therefore, it suffices to show that the class of η is not odd. If the class is odd, then there is existence of Reeb vector fields, that is, a vector field $R \in \mathfrak{X}(M)$ satisfying

$$i_R\eta = 1\,, \qquad i_R\mathrm{d}\eta = 0\,.$$

But contracting $\eta \wedge (\mathrm{d}\eta)^r$ by a Reeb vector field one obtains,

$$0 = i_R(\eta \wedge (\mathrm{d}\eta)^r) = (\mathrm{d}\eta)^r\,,$$

which is in contradiction with the assumption that $(\mathrm{d}\eta)^r \neq 0$. Thus, the class of η must be $2r$. □

Proposition 4.2.4. *Let $\eta \in \Omega^1(M)$ be a 1-form on a manifold M. Then, η has class $2r+1$ if and only if it satisfies*

$$\begin{cases} \eta \wedge (\mathrm{d}\eta)^r \neq 0\,, \\ (\mathrm{d}\eta)^{r+1} = 0\,. \end{cases} \tag{4.3}$$

Proof. Assume that η has class $2r+1$. This, again, implies that the rank of $\mathrm{d}\eta$ is $2r$, so

$$(\mathrm{d}\eta)^r \neq 0\,, \qquad (\mathrm{d}\eta)^{r+1} = 0\,,$$

by Theorem 4.1.3. If we had that $\eta \wedge (\mathrm{d}\eta)^r = 0$, by contracting this expression by a Reeb vector field we obtain

$$0 = i_R(\eta \wedge (\mathrm{d}\eta)^r) = (\mathrm{d}\eta)^r\,,$$

which is a contradiction with the fact that $\mathrm{d}\eta$ is of rank $2r$. Thus, necessarily,

$$\eta \wedge (\mathrm{d}\eta)^r \neq 0$$

as we wanted to see.

Conversely, it is clear that the assumptions (4.3) imply that $\mathrm{d}\eta$ has rank $2r$. The class, then, can only be equal to $2r$ or $2r+1$. Let us assume that the class is even. This implies the existence of Liouville vector fields, that is, there exists a vector field $\Delta \in \mathfrak{X}(M)$ satisfying

$$i_\Delta \mathrm{d}\eta = \eta\,.$$

If we contract $(\mathrm{d}\eta)^{r+1}$ by a Liouville vector field, we obtain

$$0 = i_\Delta (\mathrm{d}\eta)^{r+1} = (r+1)\eta \wedge (\mathrm{d}\eta)^r\,.$$

But, by assumption, $\eta \wedge (\mathrm{d}\eta)^r \neq 0$, so there cannot exist any Liouville vector fields. Therefore, the class of η is necessarily odd. □

With these propositions one can easily prove the existing correspondence between precontact manifolds and the class of the corresponding 1-form.

Corollary 4.2.5. *A 1-form $\eta \in \Omega^1(M)$ defines a precontact manifold (M, η) of rank $2r+1$ if and only if η has constant class $2r+1$ or $2r+2$.*

If $\eta \in \Omega^1(M)$ satisfies

$$\begin{cases} \eta \wedge (\mathrm{d}\eta)^r \neq 0\,, \\ (\mathrm{d}\eta)^{r+1} = 0\,. \end{cases}$$

then η defines a precontact manifold of rank $2r+1$ and class $2r+1$. Alternatively, if

$$\begin{cases} \eta \wedge (\mathrm{d}\eta)^{r+1} = 0\,, \\ (\mathrm{d}\eta)^{r+1} \neq 0\,. \end{cases}$$

then η defines a precontact manifold of rank $2r+1$ and class $2r+2$.

Example 4.2.6. A contact manifold (M, η) in a manifold M of dimension $2n+1$ is, in this context, a precontact manifold of rank $2n+1$. The class of η is necessarily $2n+1$, the class cannot be $2n+2$ in this case because $(\mathrm{d}\eta)^{n+1}$ is a $(2n+2)$-form in a manifold of dimension $2n+1$, and so it must be zero.

An exact symplectic manifold $(M, \mathrm{d}\theta)$, with M a manifold of dimension $2n$ and $\theta \in \Omega^1(M)$, is a precontact manifold of rank $2n-1$ and class $2n$. Indeed, because a manifold is symplectic if and only if the $(2n)$-form $(\mathrm{d}\eta)^n$ is a volume form, i.e. is nowhere-vanishing. In this case, as $\mathrm{d}\theta$ defines an isomorphism, the Liouville vector field

$$i_\Delta \mathrm{d}\theta = \theta\,,$$

is uniquely determined.

This shows that the definition of precontact manifold given here is not enough to completely determine the class of the 1-form. This is very relevant in our study, because as we have shown the class of the 1-form contains a good amount of significant information about the geometric structure. For example, the existence of Reeb vector fields is dependant on the parity of the class of the 1-form.

There exists a Darboux theorem characterizing the local expression of precontact manifolds, we refer the reader to [40] for a proof.

Theorem 4.2.7. *Let $\eta \in \Omega^1(M)$ be a nowhere-vanishing 1-form of constant class $2r+1$. Then, for all $x \in M$, there exists local coordinates $q^1, \dots, q^r, p_1, \dots, p_r, s, u_1, \dots, u_z$ (with $2r+z+1=m$) such that*

$$\eta = \mathrm{d}s - \sum_{i=1}^{r} p_i \mathrm{d}q^i\,.$$

In these coordinates, the characteristic distribution of η is given by

$$\mathcal{C} = \left\langle \left\{ \frac{\partial}{\partial u_a} \right\}_{a=1,\dots,z} \right\rangle .$$

Theorem 4.2.8. *Let $\eta \in \Omega^1(M)$ be a nowhere-vanishing 1-form of constant class $2r+2$. Then, for all $x \in M$, there exists local coordinates $q^1, \dots, q^{r+1}, p_1, \dots, p_{r+1}, u_1, \dots, u_{z-1}$ (where we have that $2r+2+z-1=m$) such that*

$$\eta = \sum_{i=1}^{r+1} p_i \mathrm{d}q^i\,.$$

In these coordinates, the characteristic distribution of η is given by

$$\mathcal{C} = \left\langle \left\{ \frac{\partial}{\partial u_a} \right\}_{a=1,\dots,z-1} \right\rangle .$$

For any nowhere-vanishing 1-form $\eta \in \Omega^1(M)$ of constant class, we have the $\mathscr{C}^\infty(M)$-module morphism

$$\begin{aligned} B\colon \mathfrak{X}(M) &\longrightarrow \Omega^1(M) \\ X &\longmapsto i_X \mathrm{d}\eta + (i_X\eta)\eta\,. \end{aligned}$$

As a matter of fact, this morphism is an isomorphism if and only if η is a contact form or $\mathrm{d}\eta$ is symplectic [2].

Proposition 4.2.9. *Let (M,η) be a precontact manifold. Then,*

$$\mathcal{C} = \operatorname{Ker}\eta \cap \operatorname{Ker}\mathrm{d}\eta = \operatorname{Ker} B = (\operatorname{Im} B)^\circ\,.$$

Proof. Let us see first that $\operatorname{Ker}\eta \cap \operatorname{Ker}\mathrm{d}\eta = \operatorname{Ker} B$. The inclusion $\operatorname{Ker}\eta \cap \operatorname{Ker}\mathrm{d}\eta \subseteq \operatorname{Ker} B$ is trivial. For the other inclusion, assume that we have $B(X)=0$, i.e.

$$B(X) = i_X\mathrm{d}\eta + (i_X\eta)\eta = 0\,.$$

If we are in the case where the class of η is odd, contracting the expression by a Reeb vector field yields $i_X\eta = 0$. If we are in the case where the class of η is even, then, by contracting the expression by a Liouville vector field, we get

$$i_\Delta i_X \mathrm{d}\eta + (i_X\eta) i_\Delta\eta = -i_X\eta = 0\,,$$

where we have used $i_\Delta \mathrm{d}\eta = \eta$ and the fact that $\Delta \in \operatorname{Ker}\eta$. Thus, in both cases, necessarily $i_X\eta = 0$, which also directly implies that $i_X\mathrm{d}\eta = 0$.

Finally, we want to see that $(\operatorname{Im} B)^\circ = \operatorname{Ker} B$. First we prove that $(\operatorname{Im} B)^\circ \supseteq \operatorname{Ker} B = \mathcal{C}$. Indeed, one easily sees that, for any vector field Y and any $X \in \mathcal{C}$, we have

$$i_X B(Y) = i_X i_Y \mathrm{d}\eta + (i_Y\eta) i_X\eta = -i_Y i_X \mathrm{d}\eta = 0\,.$$

Lastly, note that, as at every point $p \in \mathrm{T}_pM$ both subspaces have the same dimension, both distributions are necessarily equal. □

As a consequence of last proposition, we have that

$$\mathcal{C}^\circ = (\operatorname{Ker} B)^\circ = \operatorname{Im} B\,,$$

and so the class is equal to the dimension of $\operatorname{Im} B$.

Proposition 4.2.10. *Let $\eta \in \Omega^1(M)$ be a nowhere-vanishing 1-form of constant odd class $2r+1$. Then, a vector field X is a Reeb vector field if and only if $B(X) = \eta$. Hence, all Reeb vector fields are given by $R = R_0 + \Gamma$, where R_0 is a particular Reeb vector field and $\Gamma \in \mathcal{C}$.*

Proof. If R is a Reeb vector field then it is direct to check that $B(R) = \eta$. Conversely, if for some vector field X, one has that

$$i_X\mathrm{d}\eta + (i_X\eta)\eta = \eta \quad \Longrightarrow \quad i_X\mathrm{d}\eta = (1 - i_X\eta)\eta\,.$$

Contracting the expression with a Reeb vector field we obtain that, necessarily, $i_X\eta = 1$. From which it follows that $i_X\mathrm{d}\eta = 0$, and so X satisfies all conditions to be a Reeb vector field. □

Proposition 4.2.11. *Let $\eta \in \Omega^1(M)$ be a nowhere-vanishing 1-form of constant even class $2r+2$. Then, a vector field X is a Liouville vector field if and only if $B(X) = \eta$. Hence, all Liouville vector fields are given by $\Delta = \Delta_0 + \Gamma$, where Δ_0 is a particular Liouville vector field and $\Gamma \in \mathcal{C} = \operatorname{Ker}\mathrm{d}\eta$.*

Proof. If Δ is a Liouville vector field then it is direct to check that $B(\Delta) = \eta$, as the fact that $\Delta \in \operatorname{Ker}\eta$ follows immediately from its definition. Conversely, if for some vector field X, one has that

$$i_X\mathrm{d}\eta + (i_X\eta)\eta = \eta \quad \Longrightarrow \quad i_X\mathrm{d}\eta = (1 - i_X\eta)\eta\,.$$

Contracting this expression with a Liouville vector field, one gets

$$i_\Delta i_X\mathrm{d}\eta = -i_X\eta = 0\,,$$

and so, necessarily, $X \in \operatorname{Ker}\eta$. This, then, also implies that $i_X\mathrm{d}\eta = \eta$, which means that X is a Liouville vector field. □

4.3 Precontact Hamiltonian systems

Some attempts to characterize precontact structures and apply them to the study of singular contact Lagrangian functions can be found in [26, 61], and in the context of time-dependent contact mechanics, in [22]. However, these works do not address cases where the geometric structure *fails to define Reeb vector fields*. In other words, they are restricted to precontact manifolds of odd class.

This limitation arises because, until now, the contact Hamiltonian equations (3.1) have been formulated under the assumption that a Reeb vector field exists. Nevertheless, by using the equivalent equations (3.3), one can study the dynamics even in the absence of a Reeb vector field. These equations are also better suited for situations in which multiple Reeb vector fields exist.

Definition 4.3.1. Given a precontact manifold (M, η) of constant rank, and a function $H\colon M \to \mathbb{R}$, we say that the triple (M, η, H) is a *precontact Hamiltonian system*.

Given a precontact Hamiltonian system (M, η, H), its *precontact Hamiltonian equations* are the equations

$$\begin{cases} i_X(\eta \wedge \mathrm{d}\eta) = -H\mathrm{d}\eta + \mathrm{d}H \wedge \eta\,, \\ i_X\eta = -H\,. \end{cases} \tag{4.4}$$

In general, these equations define a singular system of differential equations—that is, solutions may not exist at every point, and even when they do, they may not be unique. A suitable constraint algorithm can be devised for them, for example applying the theory of *linearly singular systems* [47, 48, 52, 54].

Note that, if η has odd class and thus there is existence of Reeb vector fields, contracting the first equation of (4.4) by any Reeb vector field $R \in \mathfrak{X}(M)$ one obtains

$$i_X\mathrm{d}\eta = \mathrm{d}H - (\mathscr{L}_R H)\eta\,,$$

Hence, this implies that for there to be solutions on $x \in M$ we need to have

$$\mathscr{L}_R H(x) = \mathscr{L}_{R'} H(x)\,,$$

for any two different Reeb vector fields $R, R' \in \mathfrak{X}(M)$. By Proposition 4.2.10, we can see that this is equivalent to having the constraints

$$\mathscr{L}_\Gamma H(x) = 0\,, \tag{4.5}$$

for any $\Gamma \in \mathcal{C} = \operatorname{Ker}\eta \cap \operatorname{Ker}\mathrm{d}\eta$.

Similarly, if η has even class, contracting the first equation of (4.4) by any Liouville vector field $\Delta \in \mathfrak{X}(M)$, we obtain the constraints

$$\mathscr{L}_\Delta H = H\,. \tag{4.6}$$

Note that these last constraints *also* imply that in the even case we have $\mathscr{L}_\Gamma H(x) = 0$, for any $\Gamma \in \mathcal{C} = \operatorname{Ker}\mathrm{d}\eta$.

Let us now point out the following lemma, presented in [40].

Lemma 4.3.2. *Let $\eta \in \Omega^1(M)$ be a nowhere-vanishing 1-form of constant class $2r$ on M. Then, for any point $x \in M$, there exists a neighbourhood V of x and a function $g \in \mathscr{C}^\infty(V)$, nowhere-vanishing in V, such that*

$$\eta' = g\eta|_V$$

is of constant class $2r - 1$.

Now, recall that the codimension 1 distribution spanned by kernel of η is the same one as the one spanned by the kernel of $g\eta$, for $g \in \mathscr{C}^\infty(M)$ any nowhere-vanishing function. Also, regarding the case of precontact Hamiltonian mechanics we are interested in, it is easy to check that the contact Hamilton equations (3.3) of a precontact system (M, η, H), are the same ones as those defined by the precontact Hamiltonian system $(M, g\eta, gH)$, with $g \in \mathscr{C}^\infty(M)$ a nowhere-vanishing function. As a matter of fact, one has that:

Proposition 4.3.3. *Let (M, η, H) be a precontact Hamiltonian system. The contact Hamilton equations (3.3) defined by the precontact manifold $(M, g\eta, \widetilde{H})$, with $g \in \mathscr{C}^\infty(M)$ nowhere-vanishing and $\widetilde{H} \in \mathscr{C}^\infty(M)$, are the same as those defined by (M, η, H) if and only if $\widetilde{H} = gH$.*

Proof. First assume that $\widetilde{H} = gH$, with g nowhere-vanishing. Then, the contact Hamilton equations for $(M, g\eta, gH)$ are

$$\begin{cases} (i_X \mathrm{d}(g\eta)) \wedge g\eta = \mathrm{d}(gH) \wedge (g\eta)\,, \\ i_X(g\eta) = -gH\,. \end{cases}$$

The second equation, using that g is nowhere-vanishing, implies that $i_X\eta = -H$, which is precisely one of the contact Hamilton equations for the system (M, η, H). It only remains to see that the first contact Hamilton equation of (M, η, H) is also satisfied. For this, note that

$$(i_X \mathrm{d}(g\eta)) \wedge g\eta = (\mathrm{d}(gH)) \wedge (g\eta) \Rightarrow ((i_X \mathrm{d}g)\eta - (i_X\eta)\mathrm{d}g + g i_X \mathrm{d}\eta) \wedge g\eta = H\mathrm{d}g + g\mathrm{d}H \wedge g\eta$$

and using that $i_X\eta = -H$ we get

$$(gH)\mathrm{d}g \wedge \eta + g^2(i_X \mathrm{d}\eta \wedge \eta) = (gH)\mathrm{d}g \wedge \eta + g^2(\mathrm{d}H \wedge \eta)\,,$$

which implies, using again that g is nowhere-vanishing, that

$$i_X(\mathrm{d}\eta) \wedge \eta = \mathrm{d}H \wedge \eta\,,$$

just as we wanted to see.

Conversely, assume that $(M, g\eta, \widetilde{H})$ defines the same contact Hamilton equations as (M, η, H). Then, the second equation of $(M, g\eta, \widetilde{H})$ is

$$i_X(g\eta) = -\widetilde{H}\,,$$

which implies, assuming g is nowhere-vanishing, that

$$i_X\eta = -\frac{\widetilde{H}}{g}\,,$$

but as both systems define the same contact Hamilton equations, then, it is also true that

$$i_X\eta = -H$$

for the same vector field X. And therefore, we have that $\widetilde{H} = gH$, as we wanted to see. ☐

This motivates a deeper study of nowhere-vanishing functions that can alter the parity of the 1-form's class.

Lemma 4.3.4. *Let $\eta \in \Omega^1(M)$ be a 1-form on M and $f \in \mathscr{C}^\infty(M)$ a function. Then, we have that*

$$\mathrm{d}(e^f\eta)^n = e^{nf}(n\,\mathrm{d}f \wedge \eta \wedge (\mathrm{d}\eta)^{n-1} + (\mathrm{d}\eta)^n)\,,$$

for every $n \in \mathbb{N}$.

Proof. Let us prove this by induction. For the case $n = 1$ it is trivial as

$$\mathrm{d}(e^f\eta) = e^f(\mathrm{d}f\wedge\eta + \mathrm{d}\eta)\,.$$

Now, assuming it is true for $n-1$, for the induction step we can see that

$$\begin{aligned}\mathrm{d}(e^f\eta)^n &= \mathrm{d}(e^f\eta)^{n-1}\wedge\mathrm{d}(e^f\eta)\\ &= e^{(n-1)f}((n-1)\mathrm{d}f\wedge\eta\wedge(\mathrm{d}\eta)^{n-2} + (\mathrm{d}\eta)^{n-1})\wedge e^f(\mathrm{d}f\wedge\eta+\mathrm{d}\eta)\\ &= e^{nf}(n\,\mathrm{d}f\wedge\eta\wedge(\mathrm{d}\eta)^{n-1} + (\mathrm{d}\eta)^n)\,,\end{aligned}$$

which proves the lemma. □

From the last lemma we directly observe that

$$(e^f\eta)\wedge\mathrm{d}(e^f\eta)^n = e^{(n+1)f}(\eta\wedge(\mathrm{d}\eta)^n)\,.$$

This implies that *the rank of a precontact manifold is invariant by multiplying the 1-form by a nowhere-vanishing function*.

The next proposition uses these two last results to characterize which functions change the parity of the class of η, in terms of their Lie derivative with respect to the Liouville vector fields:

Proposition 4.3.5. *Let $\eta\in\Omega^1(M)$ be a nowhere-vanishing 1-form of class $2r+2$ on a manifold M, and $f\in\mathscr{C}^\infty(M)$ a function. Then, the 1-form $\widetilde{\eta} := e^f\eta$ is of class $2r+1$ if and only if, for all Liouville vector fields $\Delta\in\mathfrak{X}(M)$ of η, the function satisfies $\mathscr{L}_\Delta f = -1$.*

Proof. Let us first assume that $\widetilde{\eta} := e^f\eta$ is of class $2r+1$. As the 1-form η has even class, there must exist Liouville vector fields $\Delta\in\mathfrak{X}(M)$ that satisfy the equation

$$i_\Delta\mathrm{d}\eta = \eta\,.$$

If we contract $(\mathrm{d}\widetilde{\eta})^{r+1}$ by any Liouville vector field Δ of η, we obtain

$$\begin{aligned}i_\Delta(\mathrm{d}\widetilde{\eta})^{r+1} &= e^{(r+1)f}((r+1)\,i_\Delta(\mathrm{d}f\wedge\eta\wedge(\mathrm{d}\eta)^r) + i_\Delta(\mathrm{d}\eta)^{r+1})\\ &= e^{(r+1)f}(((r+1)\mathscr{L}_\Delta f)\,\eta\wedge(\mathrm{d}\eta)^r + (r+1)\,\eta\wedge(\mathrm{d}\eta)^r)\\ &= ((r+1)e^{(r+1)f}(\mathscr{L}_\Delta f + 1))\,\eta\wedge(\mathrm{d}\eta)^r\,.\end{aligned}\tag{4.7}$$

where we have used Lemma 4.3.4 in the first equality. Now, by the hypothesis of η being of class $2r+2$ and $\widetilde{\eta}$ of class $2r+1$ we have that

$$(\mathrm{d}\widetilde{\eta})^{r+1} = 0\,,\qquad \eta\wedge(\mathrm{d}\eta)^r\neq 0\,,$$

and therefore Equation (4.7) implies that

$$\mathscr{L}_\Delta f + 1 = 0\,,$$

for all Liouville vector fields Δ of η, as we wanted to see.

To see the converse first note that all the solutions $\Delta\in\mathfrak{X}(M)$ to

$$i_\Delta\mathrm{d}\eta = \eta\,,$$

can be written as $\Delta = \Delta_0 + \Gamma$, where $\Delta_0 \in \mathfrak{X}(M)$ is a particular solution and $\Gamma \in \operatorname{Ker} \mathrm{d}\eta$. Thus, if the assumption is satisfied for every Liouville vector field, one has that

$$i_{\Delta_0}\mathrm{d}f + 1 = 0\,, \quad \text{and} \quad i_{\Delta_0+\Gamma}\mathrm{d}f + 1 = i_{\Delta_0}\mathrm{d}f + i_\Gamma \mathrm{d}f + 1 = 0\,,$$

which directly implies that

$$i_\Gamma \mathrm{d}f = 0\,,$$

for all $\Gamma \in \operatorname{Ker} \mathrm{d}\eta$.

Now, let us assume that $\widetilde{\eta} := e^f\eta$ is of class $2r+2$. If $\widetilde{\eta}$ is of even class, then there must exist Liouville vector fields $\widetilde{\Delta} \in \mathfrak{X}(M)$, such that

$$i_{\widetilde{\Delta}}\mathrm{d}\widetilde{\eta} = \widetilde{\eta}\,.$$

If we develop this expression we obtain

$$i_{\widetilde{\Delta}}\mathrm{d}\widetilde{\eta} = i_{\widetilde{\Delta}}(e^f(\mathrm{d}f \wedge \eta + \mathrm{d}\eta)) = e^f((i_{\widetilde{\Delta}}\mathrm{d}f)\eta - (i_{\widetilde{\Delta}}\eta)\mathrm{d}f + i_{\widetilde{\Delta}}\mathrm{d}\eta)\,,$$

now using that $\widetilde{\Delta} \in \operatorname{Ker}\widetilde{\eta} = \operatorname{Ker}\eta$ and the fact that $\widetilde{\Delta}$ is a Liouville vector field for $\widetilde{\eta}$, we get

$$e^f((\mathscr{L}_{\widetilde{\Delta}}f)\eta + i_{\widetilde{\Delta}}\mathrm{d}\eta) = e^f\eta\,.$$

Therefore, we must have

$$(1 - \mathscr{L}_{\widetilde{\Delta}}f)\eta = i_{\widetilde{\Delta}}\mathrm{d}\eta\,.$$

If we assume that $1 - \mathscr{L}_{\widetilde{\Delta}}f \neq 0$, then we have that $\frac{\widetilde{\Delta}}{1-\mathscr{L}_{\widetilde{\Delta}}f}$ is a Liouville vector field for η. And so, by assumption

$$i_{\frac{\widetilde{\Delta}}{1-\mathscr{L}_{\widetilde{\Delta}}f}}\mathrm{d}f = -1\,,$$

this implies that

$$i_{\widetilde{\Delta}}\mathrm{d}f = i_{\widetilde{\Delta}}\mathrm{d}f - 1 \qquad \Longrightarrow \qquad 0 = -1\,,$$

which is clearly a contradiction. Thus, it must be $1 - \mathscr{L}_{\widetilde{\Delta}}f = 0$. But then $\widetilde{\Delta} \in \operatorname{Ker}\mathrm{d}\eta$, which implies (by the previous observation), that

$$\mathscr{L}_{\widetilde{\Delta}}f = 0\,,$$

necessarily, and thus arriving again to a contradiction. Therefore, we cannot have any Liouville vector fields for $\widetilde{\eta}$, which implies that $\widetilde{\eta}$ must have odd class $2r+1$. □

A similar result can be given to characterize the functions which *maintain* the odd parity of a precontact manifold, now in terms of their Lie derivative with respect to the Reeb vector fields.

Proposition 4.3.6. *Let $\eta \in \Omega^1(M)$ be a nowhere-vanishing 1-form of class $2r+1$ on a manifold M, and $f \in \mathscr{C}^\infty(M)$ a function. Then, the 1-form $\widetilde{\eta} := e^f\eta$ is of class $2r+1$ if and only if, for all Reeb vector fields $R \in \mathfrak{X}(M)$ of η, the function satisfies $(i_R\mathrm{d}f)\,\eta \wedge (\mathrm{d}\eta)^r = \mathrm{d}f \wedge (\mathrm{d}\eta)^r$.*

Proof. Assume that $e^f\eta$ has class $2r+1$. Then, necessarily, it satisfies $\mathrm{d}(e^f\eta)^{r+1} = 0$. Now, using Lemma 4.3.4, we have

$$\mathrm{d}(e^f\eta)^{r+1} = e^{(r+1)f}((r+1)\,\mathrm{d}f \wedge \eta \wedge (\mathrm{d}\eta)^r + (\mathrm{d}\eta)^{r+1}) = e^{(r+1)f}((r+1)\,\mathrm{d}f \wedge \eta \wedge (\mathrm{d}\eta)^r)\,,$$

where we have used that $(d\eta)^{r+1} = 0$, by hypothesis, in the last equality. This implies that

$$\mathrm{d}f \wedge \eta \wedge (\mathrm{d}\eta)^r = 0\,.$$

And if we contract by any Reeb vector field $R \in \mathfrak{X}(M)$ of η, we obtain

$$i_R(\mathrm{d}f \wedge \eta \wedge (\mathrm{d}\eta)^r) = (i_R\mathrm{d}f)\eta \wedge (\mathrm{d}\eta)^r - \mathrm{d}f \wedge (\mathrm{d}\eta)^r = 0\,,$$

which proves the first implication.

The converse is direct, as

$$\mathrm{d}(e^f\eta)^{r+1} = e^{(r+1)f}((r+1)\,\mathrm{d}f \wedge \eta \wedge (\mathrm{d}\eta)^r + (\mathrm{d}\eta)^{r+1}) = e^{(r+1)f}((r+1)\,\mathrm{d}f \wedge \eta \wedge (\mathrm{d}\eta)^r)\,,$$

is equal to 0, because

$$\mathrm{d}f \wedge \eta \wedge (\mathrm{d}\eta)^r = -\eta \wedge \mathrm{d}f \wedge (\mathrm{d}\eta)^r = -(i_R\mathrm{d}f)\,\eta \wedge \eta \wedge (\mathrm{d}\eta)^r = 0\,.$$

□

Example 4.3.7. Consider the manifold $\mathrm{T}\mathbb{R}\times\mathbb{R}$, with coordinates (q, v, s) and the Lagrangian $L = vs$. This Lagrangian defines the contact 1-form

$$\eta_L = \mathrm{d}s - \frac{\partial L}{\partial v}\mathrm{d}q = \mathrm{d}s - s\,\mathrm{d}q\,.$$

It is easy to see that this 1-form is of class 2, as

$$\mathrm{d}\eta_L = \mathrm{d}q \wedge \mathrm{d}s \qquad \text{and} \qquad \eta_L \wedge \mathrm{d}\eta_L = 0\,.$$

Therefore, we know that there exist Liouville vector fields for η. An arbitrary vector field $X = a\frac{\partial}{\partial q} + b\frac{\partial}{\partial v} + c\frac{\partial}{\partial s}$ is a Liouville vector field if and only if

$$i_X\mathrm{d}\eta_L = \eta_L\,,$$

which implies that

$$a\,\mathrm{d}s - c\,\mathrm{d}q = \eta_L\,.$$

Thus, the Liouville vector fields are given by $\Delta = \frac{\partial}{\partial q} + b\frac{\partial}{\partial v} + s\frac{\partial}{\partial s}$, where $b \in \mathscr{C}^\infty(M)$ is any arbitrary function.

By Proposition 4.3.5, the functions f such that the 1-form $e^f\eta_L$ has class 1 are those that satisfy

$$\frac{\partial f}{\partial q} + s\frac{\partial f}{\partial s} + 1 = 0\,, \quad \text{and} \quad \frac{\partial f}{\partial v} = 0\,.$$

Let us take, for example, $f = -q$. Then, the 1-form

$$e^{-q}\,\eta_L = e^{-q}\,\mathrm{d}s - e^{-q}s\,\mathrm{d}q$$

is of class 1. Indeed, we can find its Darboux coordinates very easily as

$$e^{-q}\eta_L = \mathrm{d}(e^{-q}s)\,.$$

5. Singular contact Lagrangian systems

The aim of this chapter is to develop tools for analysing the formalism of singular contact Lagrangian systems. To this end, we investigate the conditions under which the Lagrangian energy remains $\mathcal{F}L$-projectable, allowing for a Hamiltonian description. Also, we define and develop the properties of the evolution operator K for dissipative systems in the context of contact mechanics. We show how the main characteristics of this operator in symplectic mechanics are naturally transferred to the contact case.

5.1 Almost-regular Lagrangian functions

As in the case of ordinary mechanics, to obtain some general results in the singular case we still need to impose some weak regularity conditions on the Lagrangian functions. Namely, we say that a contact Lagrangian function L is *almost-regular* if:

- The image of the Legendre map $P_0 := \mathcal{F}L(\mathrm{T}Q\times\mathbb{R}) \subset \mathrm{T}^*Q\times\mathbb{R}$ is a closed submanifold, called the *primary contact Hamiltonian constraint submanifold*,
- The induced map $\mathcal{F}L_0 \colon \mathrm{T}Q\times\mathbb{R}\to P_0$ is a submersion and has connected fibres.

From a local point of view it is sufficient to assume that $\mathcal{F}L$ has constant rank.

As $P_0 := \mathcal{F}L(\mathrm{T}Q\times\mathbb{R})$ is a closed submanifold, it is locally defined by the vanishing of an independent set of functions $\{\phi_\mu\}_{\mu=1,\dots,m}$ with linearly independent differentials, $\mathrm{d}\phi_\mu$, at every point. We call these functions *primary contact Hamiltonian constraint functions*.

As in the symplectic case, we can apply Lemmas 2.4.1 and 2.4.2 in this context. Hence, we have that $\mathrm{d}\phi_\mu\circ\mathcal{F}L$ form a (local) reference of the vector subbundle defined by $\operatorname{Ker}{}^t\mathrm{T}(\mathcal{F}L)\subset \mathrm{T}^*(\mathrm{T}^*Q\times\mathbb{R})$.

If we consider the canonical contact 1-form η_Q in $\mathrm{T}^*Q\times\mathbb{R}$, with the functions ϕ_μ we can define, their corresponding contact Hamiltonian vector fields

$$X_{\phi_\mu} = B^{-1}(\mathrm{d}\phi_\mu) - (\mathscr{L}_R\phi_\mu + \phi_\mu)R\,.$$

And from these, we can define the vector fields

$$X_\mu = X_{\phi_\mu} + (\mathscr{L}_R\phi_\mu + \phi_\mu)R = B^{-1}(\mathrm{d}\phi_\mu)\,,$$

which, when composing with the Legendre map, $X_\mu\circ\mathcal{F}L$, generate a set of vector fields along $\mathcal{F}L$ that constitute a reference for the vector subbundle

$$B^{-1}(\operatorname{Ker}{}^t\mathrm{T}(\mathcal{F}L)) = \operatorname{Ker}({}^t\mathrm{T}(\mathcal{F}L)\circ B)\subset (\mathrm{T}Q\times\mathbb{R})\times_{\mathcal{F}L}\mathrm{T}(\mathrm{T}^*Q\times\mathbb{R})\,.$$

Also, for $P_0 \overset{j}{\hookrightarrow} \mathrm{T}^*Q\times\mathbb{R}$, the 1-forms $\mathrm{d}\phi_\mu|_{P_0}$ constitute, locally, a reference for the vector subbundle $\operatorname{Ker}{}^t\mathrm{T}(j)\subset P_0\times_j \mathrm{T}^*(\mathrm{T}^*Q\times\mathbb{R})$, and the $X_\mu|_{P_0}$ form a reference for the subbundle $F_0 = \operatorname{Ker}({}^t\mathrm{T}(j)\circ B|_{P_0})\subset P_0\times_j\mathrm{T}(\mathrm{T}^*Q\times\mathbb{R})$.

The Legendre map $\mathcal{F}L$ being a submersion is equivalent to $\operatorname{Ker}\mathrm{T}(\mathcal{F}L)\subset\mathrm{T}(\mathrm{T}Q\times\mathbb{R})$ being a vector subbundle. Consider the vector bundles $\tau_0\colon \mathrm{T}Q\times\mathbb{R}\to Q\times\mathbb{R}$ and $\pi_0\colon\mathrm{T}^*Q\times\mathbb{R}\to Q\times\mathbb{R}$. As we have that $\mathrm{T}(\tau_0)=\mathrm{T}(\pi_0)\circ\mathrm{T}(\mathcal{F}L)$, we have that $\operatorname{Ker}\mathrm{T}(\mathcal{F}L)$ is contained in the vertical subbundle $V(\mathrm{T}Q\times\mathbb{R})\subset\mathrm{T}(\mathrm{T}Q\times\mathbb{R})$.

As we are assuming that the rank of $\operatorname{Ker}\mathrm{T}(\mathcal{F}L)$ is constant, let us say without loss of generality that it is equal to m, we can find a local reference generated by m sections Γ_μ of $\mathrm{T}Q\times\mathbb{R}$, linearly independent at every point.

Let us construct such a frame using the primary contact Hamiltonian constraint functions. Recall that, as we can write $\mathrm{T}(\mathrm{T}Q\times\mathbb{R}) = (\mathrm{T}(\mathrm{T}Q)\times\mathbb{R})\otimes(\mathrm{T}Q\times\mathrm{T}\mathbb{R})$, every operation acting on tangent vectors on $\mathrm{T}Q$ can act on tangent vectors on $\mathrm{T}Q\times\mathbb{R}$. Hence, we have a natural extension of the geometric structures defined on $\mathrm{T}Q$ to $\mathrm{T}Q\times\mathbb{R}$. In particular, we can extend the map Γ discussed in Appendix A, to the map

$$\Gamma\colon (\mathrm{T}Q\times\mathbb{R})\times_{\mathcal{F}L}\mathrm{T}(\mathrm{T}^*Q\times\mathbb{R})\to\mathrm{T}(\mathrm{T}Q\times\mathbb{R})$$

which can act on Y, a vector field along $\mathcal{F}L$, as

$$\Gamma\circ Y = \mathrm{vl}_{\mathrm{T}Q\times\mathbb{R}}(\mathrm{Id}_{\mathrm{T}Q\times\mathbb{R}},\rho\circ\mathrm{T}\pi_0\circ Y)$$

to yield a vector field on $\mathrm{T}Q\times\mathbb{R}$.

One can check that for the vector fields along the Legendre map $X_\phi\circ\mathcal{F}L = B^{-1}(\mathrm{d}\phi)\circ\mathcal{F}L$, where $\phi\in\mathscr{C}^\infty(\mathrm{T}^*Q\times\mathbb{R})$, the vector field $\Gamma_\phi = \Gamma\circ X_\phi\circ\mathcal{F}L$ on $\mathrm{T}Q\times\mathbb{R}$, has local coordinates

$$\Gamma_\phi = \mathcal{F}L^*\left(\frac{\partial\phi}{\partial p}\right)\frac{\partial}{\partial v}\,.$$

With this we can prove the following result:

Proposition 5.1.1. *The vector fields $\Gamma_\mu = \Gamma\circ B^{-1}(\mathrm{d}\phi_\mu)\circ\mathcal{F}L$, constructed from the primary contact Hamiltonian constraint functions ϕ_μ, form a local reference for* $\operatorname{Ker}\mathrm{T}(\mathcal{F}L)$. *Their local expression is*

$$\Gamma_\mu = \gamma_\mu\frac{\partial}{\partial v}\,,$$

where the functions

$$\gamma_\mu = \mathcal{F}L^*\left(\frac{\partial\phi_\mu}{\partial p}\right)$$

form a basis of the kernel of the Hessian matrix $W = \left(\dfrac{\partial^2 L}{\partial v^i\partial v^j}\right)$.

Proof. By the chain rule

$$\begin{aligned}\mathrm{d}\mathcal{F}L^*(\phi) = {}&\left(\mathcal{F}L^*\left(\frac{\partial\phi}{\partial q}\right)+\mathcal{F}L^*\left(\frac{\partial\phi}{\partial p}\right)\frac{\partial^2 L}{\partial v\partial q}\right)\mathrm{d}q\\ &+\mathcal{F}L^*\left(\frac{\partial\phi}{\partial p}\right)W\mathrm{d}v+\left(\mathcal{F}L^*\left(\frac{\partial\phi}{\partial s}\right)+\mathcal{F}L^*\left(\frac{\partial\phi}{\partial p}\right)\frac{\partial^2 L}{\partial v\partial s}\right)\mathrm{d}s\,,\end{aligned}$$

and if ϕ_μ is a primary Hamiltonian constraint function then $\mathrm{d}\mathcal{F}L^*(\phi_\mu)=0$ necessarily, which implies that $\gamma_\mu W = 0$. Now, locally, we check that

$$\mathrm{T}(\mathcal{F}L)\circ\Gamma_\mu(q,v,s) = \left(q,\frac{\partial L}{\partial v},s,0,\gamma_\mu W,0\right),$$

and so, $\Gamma_\mu\in\operatorname{Ker}\mathrm{T}(\mathcal{F}L)$. Also, the m vector fields are linearly independent because the γ_μ also are, as a consequence of the linear independency of the $\mathrm{d}\phi_\mu$. Hence, they form a reference for $\operatorname{Ker}\mathrm{T}(\mathcal{F}L)$. $\square$

This allows us to prove the most important result for the characterization of a Hamiltonian formalism in this context.

Proposition 5.1.2. *If the Legendre map $\mathcal{F}L$ is a submersion, then the contact Lagrangian energy function E_L is locally projectable, that is, there exists a function H, such that (locally) $E_L = H \circ \mathcal{F}L$.*

Proof. A necessary and sufficient condition for the existence of H is that $\Gamma_\mu(E_L) = 0$, i.e. if a function vanishes by $\operatorname{Ker} \mathrm{T}(\mathcal{F}L)$ then it is $\mathcal{F}L$-projectable. We have

$$\Gamma_\mu(E_L) = \gamma_\mu \frac{\partial}{\partial v}\left(v\frac{\partial L}{\partial v} - L\right) = \gamma_\mu(Wv) = 0\,,$$

and so the existence of H is guaranteed. $\square$

If the Lagrangian function L is almost-regular, then E_L is *globally* $\mathcal{F}L$-projectable at P_0, that is, there exists a *unique* function $H_0 \colon P_0 \to \mathbb{R}$, the *contact Hamiltonian* function, such that $\mathcal{F}L^*(H_0) = E_L$. Also, as P_0 is assumed to be closed the function H_0 can be extended to a function H defined on $\mathrm{T}^*Q \times \mathbb{R}$. Note that for a local study it is enough to consider that the Legendre map $\mathcal{F}L$ is a submersion.

With a Hamiltonian function defined on P_0, we can consider the precontact 1-form

$$\eta_0 = j^*(\eta_Q)\,,$$

to obtain the precontact Hamiltonian system (P_0, ω_0, H_0). The solutions and constraints that appear when solving the precontact Hamiltonian equations for this system can be related to those that appear for the precontact system $(\mathrm{T}Q \times \mathbb{R}, \eta_L, E_L)$ by means of the evolution operator K, which we present in the following section.

5.2 Evolution operator for contact systems

In this section we present an intrinsic characterization of the evolution operator$K$ for contact systems, along with some of its most relevant properties. We follow the structure of Section 2.5. First, we define the K evolution operator in the contact mechanics, interpreting it as a vector field along the Legendre map. Then, we prove that the corresponding analogue properties still hold in this framework.

5.2.1 Intrinsic characterization of the evolution operator

The contact evolution operator K can be intrinsically characterized in a way that closely parallels the standard case, presented in Section 2.5.1. Specifically, it is defined as the unique vector field along the contact Legendre map (3.2.5) that satisfies three key conditions: the second-order condition and the two dynamical conditions.

So, first of all let us write the condition of having a vector field along the contact Legendre map. That is, we ask for the following diagram

$$\begin{tikzcd}
 & \mathrm{T}(\mathrm{T}^*Q \times \mathbb{R}) \arrow[d, "\tau_{\mathrm{T}^*Q\times\mathbb{R}}"] \\
\mathrm{T}Q \times \mathbb{R} \arrow[ur, "K"] \arrow[r, "\mathcal{F}L"'] & \mathrm{T}^*Q \times \mathbb{R}
\end{tikzcd}$$

to be commutative. Or, in other words, that $K\colon \mathrm{T}Q\times\mathbb{R}\to\mathrm{T}(\mathrm{T}^*Q\times\mathbb{R})$ satisfies

$$\tau_{\mathrm{T}^*Q\times\mathbb{R}}\circ K=\mathcal{F}L\,.$$

This equation is sometimes referred to as the structural equation and it implies that, in local coordinates, K is expressed as

$$K(q,v,s)=\left(q,\frac{\partial L}{\partial v},s,a(q,v,s),b(q,v,s),c(q,v,s)\right)\,,$$

or, alternatively, as

$$K(q,v)=a^i(q,v,s)\frac{\partial}{\partial q^i}\bigg|_{\mathcal{F}L(q,v,s)}+b^i(q,v,s)\frac{\partial}{\partial p_i}\bigg|_{\mathcal{F}L(q,v,s)}+c(q,v,s)\frac{\partial}{\partial s}\bigg|_{\mathcal{F}L(q,v,s)}\,,$$

where a^i, b^i and c are yet to be determined.

Now, let us give the so-called *second-order condition* for the K contact operator. It is

$$\mathrm{T}\pi_1\circ K=\mathrm{Id}_{\mathrm{T}Q}\,,\tag{5.1}$$

where $\pi_1\colon \mathrm{T}^*Q\times\mathbb{R}\to Q$ is the canonical projection. If we write this in coordinates, we find that

$$\mathrm{T}\pi_1(q,p,s,v,u,z)=(q,v)\,,$$

and so,

$$\mathrm{T}\pi_1\circ K=(q,a^i)\,.$$

Thus, Equation (5.1) determines that, in coordinates, we have $a^i=v^i$.

The last step to completely characterize the contact K operator is through the so-called *dynamical conditions*. They are

$$\begin{cases}\mathcal{F}L^*(i_K(\mathrm{d}\eta_Q\circ\mathcal{F}L))=\mathrm{d}E_L+\dfrac{\partial L}{\partial s}\eta_L\,,\\ i_K(\eta_Q\circ\mathcal{F}L)=-E_L\,,\end{cases}$$

where η_Q is the canonical contact form on $\mathrm{T}^*Q\times\mathbb{R}$.

In coordinates, these conditions define the functions b^i and c. Let us see this, by writing in local coordinates all the expressions involved. First, we have that

$$i_K(\mathrm{d}\eta_Q\circ\mathcal{F}L)=-b^i\mathrm{d}q^i\big|_{\mathcal{F}L(q,v,s)}+v^i\mathrm{d}p_i\big|_{\mathcal{F}L(q,v,s)}\,,$$

and therefore,

$$\mathcal{F}L^*(i_K(\mathrm{d}\eta_Q\circ\mathcal{F}L))=\left(v^j\frac{\partial^2L}{\partial v^j\partial q^i}-b^i\right)\mathrm{d}q^i\big|_{\mathcal{F}L}+v^i\frac{\partial^2L}{\partial v^i\partial v^j}\mathrm{d}v^i\big|_{\mathcal{F}L}+v^i\frac{\partial^2L}{\partial v^i\partial s}\mathrm{d}s\big|_{\mathcal{F}L}\,.$$

On the right-hand side we have

$$\mathrm{d}E_L+\frac{\partial L}{\partial s}\eta_L=\left(v^j\frac{\partial^2L}{\partial v^j\partial q^i}-\frac{\partial L}{\partial q^i}-\frac{\partial L}{\partial s}\frac{\partial L}{\partial v^i}\right)\mathrm{d}q^i\big|_{\mathcal{F}L}+v^i\frac{\partial^2L}{\partial v^i\partial v^j}\mathrm{d}v^i\big|_{\mathcal{F}L}+v^i\frac{\partial^2L}{\partial v^i\partial s}\mathrm{d}s\big|_{\mathcal{F}L}\,,$$

and so, equating both expressions, we obtain

$$b^i = \frac{\partial L}{\partial q^i} + \frac{\partial L}{\partial s}\frac{\partial L}{\partial v^i}\,.$$

The second dynamical condition, taking into account that

$$i_K(\eta_Q \circ \mathcal{F}L) = c(q,v,s) - v^i\frac{\partial L}{\partial v^i}\,,$$

implies directly that $c = L$.

Thus, we have proved that with these three conditions, and the fact that the contact K operator is a vector field along the Legendre map, the evolution operator $K\colon \mathrm{T}Q\times\mathbb{R} \to \mathrm{T}(\mathrm{T}^*Q\times\mathbb{R})$ is *fully and intrinsically determined*. Also, we have seen that its coordinate expression is

$$K(q,v,s) = v^i\left(\frac{\partial}{\partial q^i}\circ\mathcal{F}L\right) + \left(\frac{\partial L}{\partial q^i} + \frac{\partial L}{\partial s}\frac{\partial L}{\partial v^i}\right)\left(\frac{\partial}{\partial p_i}\circ\mathcal{F}L\right) + L\left(\frac{\partial}{\partial s}\circ\mathcal{F}L\right)\,.$$

Additionally, the evolution operator defines a map $K\colon \mathscr{C}^\infty(\mathrm{T}^*Q\times\mathbb{R}) \to \mathscr{C}^\infty(\mathrm{T}Q\times\mathbb{R})$ that takes functions defined on the contact Hamiltonian formalism and gives the time derivate of the function in the contact Lagrangian formalism. In local coordinates, this derivation is given by

$$(K\cdot f)(q,v,s) = v^i\mathcal{F}L^*\left(\frac{\partial f}{\partial q^i}\right) + \left(\frac{\partial L}{\partial q^i} + \frac{\partial L}{\partial s}\frac{\partial L}{\partial v^i}\right)\mathcal{F}L^*\left(\frac{\partial f}{\partial p_i}\right) + L\,\mathcal{F}L^*\left(\frac{\partial f}{\partial s}\right)\,.$$

Note that, we can also write $(K\cdot f)\in\mathscr{C}^\infty(\mathrm{T}Q\times\mathbb{R})$ as the function defined by

$$(K\cdot f)(q,v,s) = \langle \mathrm{d}f(\mathcal{F}L(q,v,s)), K(q,v,s)\rangle\,. \tag{5.2}$$

The contact evolution operator can also be characterized with one dynamical condition, with the use of canonical isomorphism $B\colon \mathrm{T}(\mathrm{T}^*Q\times\mathbb{R}) \to \mathrm{T}^*(\mathrm{T}^*Q\times\mathbb{R})$ defined by the canonical contact 1-form on $\mathrm{T}^*Q\times\mathbb{R}$.

Proposition 5.2.1. *The contact evolution operator K can be equivalently characterized as the unique the vector field along the Legendre transformation $\mathcal{F}L$ satisfying*

$$\mathrm{T}\pi_1\circ K = \mathrm{Id}_{\mathrm{T}Q}\,,$$

and

$$\mathcal{F}L^*(B\circ K) = \mathrm{d}E_L + \left(\frac{\partial L}{\partial s} - E_L\right)\eta_L\,.$$

Proof. We can prove this in coordinates in a similar way as before. The first equation is the second-order condition, and as we saw before it implies $a^i = v^i$. For the second equation, on the left-hand side we have

$$\begin{aligned}\mathcal{F}L^*(B\circ K) = {}^t\mathrm{T}\mathcal{F}L\circ(i_K(\mathrm{d}\eta_Q\circ\mathcal{F}L) + (i_K(\eta_Q\circ\mathcal{F}L))(\eta_Q\circ FL)) = \\ \left(v^j\frac{\partial^2 L}{\partial v^j\partial q^i} - b^i - c\frac{\partial L}{\partial v^i} + v^j\frac{\partial L}{\partial v^j}\frac{\partial L}{\partial v^i}\right)\mathrm{d}q^i\big|_{\mathcal{F}L} \\ + v^i\frac{\partial^2 L}{\partial v^i\partial v^j}\mathrm{d}v^i\big|_{\mathcal{F}L} + \left(v^i\left(\frac{\partial^2 L}{\partial v^i\partial s} - \frac{\partial L}{\partial v^i}\right) + c\right)\mathrm{d}s\big|_{\mathcal{F}L}\,,\end{aligned}$$

and on the right-hand side

$$
\begin{aligned}
&\mathrm{d}E_L + \left(\frac{\partial L}{\partial s} - E_L\right)\eta_L = \\
&\left(v^j \frac{\partial^2 L}{\partial v^j \partial q^i} - \frac{\partial L}{\partial q^i} - \frac{\partial L}{\partial s}\frac{\partial L}{\partial v^i} + v^j \frac{\partial L}{\partial v^j}\frac{\partial L}{\partial v^i} - L\frac{\partial L}{\partial v^i}\right)\mathrm{d}q^i\big|_{\mathcal{F}L} \\
&+ v^i \frac{\partial^2 L}{\partial v^i \partial v^j}\mathrm{d}v^i\big|_{\mathcal{F}L} + \left(v^i\left(\frac{\partial^2 L}{\partial v^i \partial s} - \frac{\partial L}{\partial v^i}\right) + L\right)\mathrm{d}s\big|_{\mathcal{F}L},
\end{aligned}
$$

equating both sides directly yields the local expression of the contact evolution operator. □

5.2.2 Properties relating the Lagrangian and Hamiltonian formulations

We present here some relevant properties of the contact evolution operator K. Most of the following propositions are analogous to those we presented, for the symplectic case, in Section 2.5.2.

Proposition 5.2.2. *Let $\xi\colon I \to \mathrm{T}Q \times \mathbb{R}$ be a path, and $\dot{\xi}\colon I \to \mathrm{T}(\mathrm{T}Q \times \mathbb{R})$ its canonical lift. Then, ξ is a solution of the generalized Euler–Lagrange equations, for a given Lagrangian L, if and only if*

$$\mathrm{T}(\mathcal{F}L) \circ \dot{\xi} = K \circ \xi\,.$$

Proof. It is enough to see this in local canonical coordinates. Assume that the path is given by $\xi = (q, v, s)$. Its canonical lift is, then, $\dot{\xi} = (q, v, s, \dot{q}, \dot{v}, \dot{s})$. Thus, we have

$$\mathrm{T}(\mathcal{F}L) \circ \dot{\xi} = \left(q, \frac{\partial L}{\partial v}, s, \dot{q}, \dot{q}\frac{\partial^2 L}{\partial v \partial q} + \dot{v}\frac{\partial^2 L}{\partial v \partial v} + \dot{s}\frac{\partial^2 L}{\partial v \partial s}, \dot{s}\right),$$

and

$$K \circ \xi = \left(q, \frac{\partial L}{\partial v}, s, v, \frac{\partial L}{\partial q} + \frac{\partial L}{\partial s}\frac{\partial L}{\partial v}, L\right).$$

Equating both expressions, this yields

$$
\begin{cases}
\dot{q} = v\,, \\
v\dfrac{\partial^2 L}{\partial v \partial q} + \dot{v}\dfrac{\partial^2 L}{\partial v \partial v} + \dot{s}\dfrac{\partial^2 L}{\partial v \partial s} = \dfrac{\partial L}{\partial q} + \dfrac{\partial L}{\partial s}\dfrac{\partial L}{\partial v}\,, \\
\dot{s} = L\,,
\end{cases}
$$

which are precisely the second-order condition and the generalized Euler–Lagrange equations. □

Note that, if a path $\xi\colon I \to \mathrm{T}Q \times \mathbb{R}$ satisfies

$$\mathrm{T}(\mathcal{F}L) \circ \dot{\xi} = K \circ \xi$$

then, necessarily, it can be obtained as the prolongation of a path $\zeta\colon I \to Q \times \mathbb{R}$. Therefore, also in this formalism, the equation of motion defined by the evolution operator K incorporates the second-order condition independently of the regularity of the Lagrangian function.

A solution to the generalized Euler–Lagrange equations satisfies $\dot{\xi} = X_L \circ \xi$, with X_L a second-order Lagrangian vector field defined on an appropriate submanifold of $\mathrm{T}Q \times \mathbb{R}$. Thus, an immediate consequence of the last proposition is that we can write

$$\mathrm{T}(\mathcal{F}L) \circ X_L \circ \xi = K \circ \xi\,.$$

And note that, if S_f is the final constraint submanifold of $\mathrm{T}Q \times \mathbb{R}$, as we have solutions at every point of this submanifold, we can write

$$K|_{S_f} = \mathrm{T}(\mathcal{F}L) \circ X_L|_{S_f} .$$

Recall that, also in this contact case, if the Lagrangian is regular then the Legendre transformation is a local diffeomorphism, and the Lagrangian vector field X_L is uniquely determined. Therefore, we directly obtain

$$X_L = \mathrm{T}(\mathcal{F}L^{-1}) \circ K .$$

Proposition 5.2.2 relates contact Lagrangian vector fields and the evolution operator K. Now, with the following proposition, we relate contact Hamiltonian vector fields and the evolution operator K.

Proposition 5.2.3. *Let $\psi\colon I \to \mathrm{T}^*Q \times \mathbb{R}$ be a path on the extended cotangent bundle, and let $\dot{\psi}\colon I \to \mathrm{T}(\mathrm{T}^*Q \times \mathbb{R})$ be its canonical lift. Then, ψ is a solution to the Herglotz–Dirac equations for L if and only if*

$$\dot{\psi} = K \circ \rho \circ \mathrm{T}(\pi_0) \circ \dot{\psi} ,$$

where $\pi_0\colon \mathrm{T}^*Q \times \mathbb{R} \to Q \times \mathbb{R}$ and $\rho\colon \mathrm{T}(Q \times \mathbb{R}) \to \mathrm{T}Q \times \mathbb{R}$ are the canonical projections.

Proof. It is enough to explicitly write both sides of the equation in coordinates. If $\psi = (q, p, s)$, then its canonical lift is expressed as $\dot{\psi} = (q, p, s, \dot{q}, \dot{p}, \dot{s})$. Therefore,

$$K \circ \rho \circ \mathrm{T}(\pi_0) \circ \dot{\psi} = \left(q, \frac{\partial L}{\partial v}, s, \dot{q}, \frac{\partial L}{\partial q} + \frac{\partial L}{\partial v}\frac{\partial L}{\partial s}, L \right) .$$

Equating both sides yields

$$\begin{cases} \dot{s} = L , \\ p = \dfrac{\partial L}{\partial v} , \\ \dot{p} = \dfrac{\partial L}{\partial q} + \dfrac{\partial L}{\partial v}\dfrac{\partial L}{\partial s} , \end{cases}$$

which are precisely the Herglotz–Dirac equations for L. □

Note that, if there exists a solution to Hamilton's equation $\psi\colon I \to \mathrm{T}^*Q \times \mathbb{R}$, then it can be expressed as

$$\psi = \mathcal{F}L \circ \xi$$

where $\xi\colon I \to \mathrm{T}Q \times \mathbb{R}$ solves the generalized Euler–Lagrange equations, and is obtained as the prolongation of the projected path $(\pi_0 \circ \psi)$. Then, using this and Proposition 5.2.2, we have that

$$\dot{\psi} = \mathrm{T}(\mathcal{F}L) \circ \dot{\xi} = K \circ \xi = K \circ \rho \circ \mathrm{T}(\pi_0) \circ \dot{\psi}.$$

As K is an embedding in this formalism also, the solutions to both the generalized Euler–Lagrange and the Herglotz–Dirac equations are in bijection. The map $\xi \mapsto \mathcal{F}L \circ \xi$ sends solutions to solutions, with inverse $\psi \mapsto \rho \circ \mathrm{T}(\pi_0) \circ \dot{\psi}$.

Now, suppose that we have a Hamiltonian vector field X_H, defined on the image of the Legendre transformation $\mathcal{F}L(\mathrm{T}Q \times \mathbb{R}) \subseteq \mathrm{T}^*Q \times \mathbb{R}$, for the Hamiltonian formalism associated to the contact

Lagrangian system $(\mathrm{T}Q \times \mathbb{R}, L)$. Then, the solutions to Hamilton's equations can be written as its integral curves $\dot{\psi} = X_H \circ \psi$, and we can write

$$K \circ \xi = \dot{\psi} = X_H \circ \psi = X_H \circ \mathcal{F}L \circ \xi \,. \tag{5.3}$$

This, in the final constraint submanifold S_f, gives the relation

$$K|_{S_f} = X_H \circ \mathcal{F}L|_{S_f} \,.$$

And, if the Lagrangian is regular, then we have

$$X_H = K \circ \mathcal{F}L^{-1} \,.$$

Lastly, assume that we have two solutions ξ and ψ of, respectively, the generalized Euler–Lagrange equations and the Herglotz–Dirac equations which are related, i.e. $\psi = \mathcal{F}L \circ \xi$ and so $\xi = \rho \circ \mathrm{T}(\pi_0) \circ \dot{\psi}$. Then, for a given function $f \in \mathscr{C}^\infty(\mathrm{T}^*Q \times \mathbb{R})$, the following holds:

$$\frac{\mathrm{d}}{\mathrm{d}t}(f \circ \psi) = \langle \mathrm{d}f \circ \psi, \dot{\psi} \rangle = \langle \mathrm{d}f \circ (\mathcal{F}L \circ \xi), K \circ \xi \rangle = (K \cdot f) \circ \xi \,,$$

where Equations (5.2) and (5.3) have been used.

This directly yields the following result.

Corollary 5.2.4. *Suppose that $f \in \mathscr{C}^\infty(\mathrm{T}^*Q \times \mathbb{R})$ is a Hamiltonian constant of motion, such as a Hamiltonian constraint. Then, $(K \cdot f)$ is a Lagrangian constraint.*

5.3 Examples

This section is devoted to the study of a couple of interesting examples. In both cases we take singular Lagrangian functions from the literature, and add to them a dissipative term. We apply a variation of Dirac's theory to determine their constraint structure.

5.3.1 The conformal particle

Let us consider the Lagrangian of the conformal particle [56, 72], with an added damping coefficient γ. Namely, we consider

$$L = \frac{1}{2} g_{ij}(v^i v^j - \lambda x^i x^j) - \gamma s \,,$$

in the phase space $\mathrm{T}(\mathbb{R}^n \times \mathbb{R}) \times \mathbb{R}$, with natural coordinates $(x^i, \lambda; v^i, w)$, where $\mathbb{R}^n$ is equipped with an indefinite constant metric g. The Legendre map is given by

$$\mathcal{F}L(x, \lambda; v, w; s) = (x,\ \lambda;\ p_i = g_{ij} v^j,\ \pi = 0;\ s) \,,$$

defining a submanifold $P_0 \subset \mathrm{T}^*(\mathbb{R}^n \times \mathbb{R}) \times \mathbb{R}$, which is determined by the vanishing of the Hamiltonian constraint $\phi_0 = \pi$.

The Lagrangian energy is

$$E_L = \frac{1}{2} g_{ij}(v^i v^j + \lambda x^i x^j) + \gamma s \,,$$

and therefore, a suitable Hamiltonian function is

$$H = \frac{1}{2}(g^{ij} p_i p_j + \lambda g_{ij} x^i x^j) + \gamma s \,.$$

Recall that any other Hamiltonian of the form $H' = H + f\phi_0$, with f an arbitrary function, would also work, as all these Hamiltonians are equal on the first constraint submanifold P_0.

The canonical contact form η_Q defined on $\mathrm{T}^*(\mathbb{R}^n \times \mathbb{R}) \times \mathbb{R}$, defines a unique contact Hamiltonian vector field X_H through the contact Hamiltonian equations (3.1). It is straightforward to check in coordinates that

$$X_H = g^{ij}p_j \frac{\partial}{\partial x^i} - (\lambda g_{ij}x^j + \gamma p_i)\frac{\partial}{\partial p_i} - \left(\frac{1}{2}g_{ij}x^ix^j + \gamma\pi\right)\frac{\partial}{\partial \pi} + \left(\frac{1}{2}\left(g^{ij}p_ip_j - \lambda g_{ij}x^ix^j\right) - \gamma s\right)\frac{\partial}{\partial s}\,.$$

With this, we can compute the following constraints. We get the constraint

$$(X_H \cdot \phi_0) = -\frac{1}{2}g_{ij}x^ix^j - \gamma\pi\,.$$

which in P_0 is equivalent to the constraint $\phi_1 = -\frac{1}{2}g_{ij}x^ix^j$.

If we keep on going, we obtain

$$(X_H \cdot \phi_1) = -p_ix^i$$

and hence $\phi_2 = -p_ix^i$ is another constraint for the system. This constraint yields,

$$(X \cdot \phi_2) = -g^{ij}p_ip_j + \lambda g_{ij}x^ix^j + \gamma p_ix^i\,,$$

which is equivalent to the constraint $\phi_3 = -g^{ij}p_ip_j$. The constraint algorithm ends here, as

$$(X_H \cdot \phi_3) = 2(-p_ix^i + \gamma g^{ij}p_ip_j)\,.$$

For this Lagrangian, the evolution operator K is given by

$$K = v^i\left(\frac{\partial}{\partial x^i}\circ\mathcal{F}L\right) + w\left(\frac{\partial}{\partial\lambda}\circ\mathcal{F}L\right) - (\lambda g_{ij}x^j + \gamma g_{ij}v^j)\left(\frac{\partial}{\partial p_i}\circ\mathcal{F}L\right) - \frac{1}{2}g_{ij}x^ix^j\left(\frac{\partial}{\partial\pi}\circ\mathcal{F}L\right) + L\left(\frac{\partial}{\partial s}\circ\mathcal{F}L\right)\,.$$

By Proposition 5.2.4, applying the K operator to the Hamiltonian constraints we obtain Lagrangian constraint functions. Namely, we obtain

$$\begin{cases} (K\cdot\phi_0) = -\dfrac{1}{2}g_{ij}x^ix^j\,, \\ (K\cdot\phi_1) = -g_{ij}v^ix^j\,, \\ (K\cdot\phi_2) = \lambda g_{ij}x^ix^j + \gamma g_{ij}x^iv^j - g_{ij}v^iv^j\,, \\ (K\cdot\phi_3) = 2(\lambda g_{ij}x^iv^j + \gamma g_{ij}v^iv^j)\,. \end{cases}$$

And therefore, we have obtained three Lagrangian constraints, because the last one can be rewritten in terms of the first 3, which are equivalent to

$$\begin{cases} \chi_1 = -\dfrac{1}{2}g_{ij}x^ix^j\,, \\ \chi_2 = -g_{ij}v^ix^j\,, \\ \chi_3 = -g_{ij}v^iv^j\,. \end{cases}$$

These Lagrangian constraints can be found, alternatively, using the precontact Hamiltonian equations (3.3). Indeed, the Lagrangian 1-form is

$$\eta_L = \mathrm{d}s - g_{ij}v^j\mathrm{d}x^i\,,$$

and therefore it defines a precontact manifold of rank $2n+1$ on $\mathrm{T}(\mathbb{R}^n\times\mathbb{R})\times\mathbb{R}$. The class of η_L is also equal to $2n+1$, as its characteristic distribution is given by

$$\mathcal{C} = \left\langle \left\{ \frac{\partial}{\partial\lambda}, \frac{\partial}{\partial w} \right\} \right\rangle .$$

The precontact Hamiltonian equations, along with the added second-order condition, yield directly after some calculations the Lagrangian constraint χ_1 and the Lagrangian vector fields

$$X_L = v^i\frac{\partial}{\partial x^i} + w\frac{\partial}{\partial\lambda} - (\lambda g_{ij}x^j + \gamma g_{ij}v^j)\frac{\partial}{\partial v^i} + f\frac{\partial}{\partial w} + L\frac{\partial}{\partial s}\,,$$

where f is an arbitrary function. We can calculate the Lie derivative of X_L with χ_1 to obtain the constraint χ_2, and again apply X_L to χ_2 to obtain χ_3. We do not obtain any more constraints after this.

One can also obtain the constraint χ_1 applying (4.5), as

$$i_{\frac{\partial}{\partial\lambda}}\mathrm{d}E_L = 0 \quad\Rightarrow\quad \frac{1}{2}x^2 = 0\,.$$

5.3.2 Cawley's Lagrangian with dissipation

Cawley's Lagrangian is an academic model based introduced by R. Cawley to study some features of singular Lagrangians in Dirac's theory of constraint systems [19]. In this example we introduce a velocity-dependent dissipation term to that Lagrangian. Consider the manifold $\mathrm{T}\mathbb{R}^3\times\mathbb{R}$ with canonical coordinates $(x,y,z;v_x,v_y,v_z;s)$ and the Lagrangian function

$$L = v_xv_z + \frac{1}{2}yz^2 - \gamma s v_y\,.$$

where γ is a non-zero damping coefficient. The Legendre map is

$$\mathcal{F}L\colon (x,y,z;v_x,v_y,v_z;s) \longmapsto (x,y,z;\, p_x = v_z,\; p_y = -\gamma s,\; p_z = v_x;\; s)$$

The first Hamiltonian constraint is $\phi_0 = p_y + \gamma s$.

The Lagrangian energy is

$$E_L = v_xv_z - \frac{1}{2}yz^2\,,$$

therefore we can take as a Hamiltonian function

$$H = p_xp_z - \frac{1}{2}yz^2\,.$$

Now, we can compute the corresponding Hamiltonian vector fields, with respect to the precontact structured defined on $P_0 := \mathcal{F}L(\mathrm{T}Q\times\mathbb{R})$ by taking $\eta_0 := j^*(\eta)$, where $j\colon P_0 \hookrightarrow \mathrm{T}^*Q\times\mathbb{R}$. In coordinates, we have

$$\eta_0 = \mathrm{d}s - p_x\mathrm{d}x + \gamma s\mathrm{d}y - p_z\mathrm{d}z$$

Note that this 1-form has even class equal to 6, and so it defines a precontact manifold of rank 5, this is because the characteristic distribution is spanned by $\frac{\partial}{\partial p_y}$. If we apply the precontact Hamiltonian equations 3.3, we obtain:

$$X_H = p_z\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + p_x\frac{\partial}{\partial z} - \gamma b p_x\frac{\partial}{\partial p_x} + c\frac{\partial}{\partial p_y} + (yz - \gamma b p_z)\frac{\partial}{\partial p_z} + \left(p_x p_z + \frac{1}{2}yz^2 - \gamma bs\right)\frac{\partial}{\partial s}\,.$$

where b and c are arbitrary functions to be determined. We also obtain as a necessary condition the constraint

$$\phi_1 \coloneqq \frac{1}{2}z^2 + \gamma\left(p_x p_z + \frac{1}{2}yz^2\right)\,.$$

Now, demanding the tangency to the first constraint submanifold

$$(X_H \cdot \phi_0) = c + \gamma\left(p_x p_z + \frac{1}{2}yz^2 - \gamma bs\right) = 0$$

we obtain an expression for c, in terms of the function b. Again, the same can be done for the constraint ϕ_1, and the condition $(X_H \cdot \phi_1) = 0$ will determine the function b. The constraint algorithm ends here.

The evolution operator is given here by

$$\begin{aligned} K =& v_x\frac{\partial}{\partial x}\Big|_{\mathcal{F}L} + v_y\frac{\partial}{\partial y}\Big|_{\mathcal{F}L} + v_z\frac{\partial}{\partial z}\Big|_{\mathcal{F}L} - \gamma v_y v_z\frac{\partial}{\partial p_x}\Big|_{\mathcal{F}L} \\ &+ \left(\frac{1}{2}z^2 + \gamma^2 s v_y\right)\frac{\partial}{\partial p_y}\Big|_{\mathcal{F}L} + (yz - \gamma v_x v_y)\frac{\partial}{\partial p_z}\Big|_{\mathcal{F}L} + L\frac{\partial}{\partial s}\Big|_{\mathcal{F}L}\,. \end{aligned}$$

We can apply the evolution operator to the Hamiltonian constraints to obtain Lagrangian ones. Namely, we obtain the following constraints

$$\begin{cases} \chi_1 \coloneqq (K \cdot \phi_0) = \frac{1}{2}z^2 + \gamma\left(v_x v_z + \frac{1}{2}yz^2\right)\,, \\ \chi_2 \coloneqq (K \cdot \phi_1) = z v_z(1 + 2\gamma y) + \gamma v_y\left(\frac{1}{2}z^2 - 2\gamma v_x v_z\right)\,. \end{cases}$$

One can check that the Lagrangian forms in this case are

$$\eta_L = \mathrm{d}s - v_z\mathrm{d}x + \gamma s\mathrm{d}y - v_x\mathrm{d}z\,,$$

and

$$\mathrm{d}\eta_L = \mathrm{d}x \wedge \mathrm{d}v_z + \mathrm{d}z \wedge \mathrm{d}v_x + \gamma\mathrm{d}s \wedge \mathrm{d}y\,.$$

The characteristic distribution is

$$\mathcal{C} = \left\langle\left\{\frac{\partial}{\partial v_y}\right\}\right\rangle\,,$$

and therefore the 1-form η_L defines a precontact manifold of rank 5 and class 6. As the class is even, we have existence of Liouville vector fields Δ. They are given by

$$\Delta = -\frac{1}{\gamma}\frac{\partial}{\partial y} + v_x\frac{\partial}{\partial v_x} + f\frac{\partial}{\partial v_y} + v_z\frac{\partial}{\partial v_z} + s\frac{\partial}{\partial s}\,,$$

where f is any arbitrary function. One can compute (4.6),

$$i_\Delta \mathrm{d}E_L = E_L \quad \Rightarrow \quad \frac{1}{2}z^2 + \gamma\left(v_x v_y + \frac{1}{2}yz^2\right) = 0\,,$$

to obtain the constraint χ_1.

Applying the precontact Hamiltonian equations (3.3), along with the second order condition, to the system $(\mathrm{T}\mathbb{R}^3 \times \mathbb{R}, \eta_L, E_L)$, one can check after some calculations that we obtain also the constraint χ_1 as a condition for there to be solutions, and that the Lagrangian vector fields are of the form

$$X_L = v_x\frac{\partial}{\partial x} + v_y\frac{\partial}{\partial y} + v_z\frac{\partial}{\partial z} + (yz - \gamma v_y v_x)\frac{\partial}{\partial v_x} + g\frac{\partial}{\partial v_y} - \gamma v_y v_z\frac{\partial}{\partial v_z} + L\frac{\partial}{\partial s}\,,$$

where g is an arbitrary function. One can check that $(X_L \cdot \chi_1) = \chi_2$. It is clear that χ_2 is not $\mathcal{F}L$-projectable, the tangency condition $(X_L \cdot \chi_2) = 0$ determines the function g.

6. Conclusions and outlook

In this thesis we have studied several geometric structures underlying singular contact Lagrangian systems. To this end, we first reviewed the fundamental concepts of geometric mechanics, placing special emphasis on singular Lagrangians and the properties and characterization of the evolution operator K. We also reviewed the basic facts of contact geometry and contact Hamiltonian systems, for which we obtained alternative formulations of the dynamical equations that do not rely on the Reeb vector field.

We reviewed the concept of class of a differential form. This helped us study the notion of precontact manifold. We analysed the conditions under which Reeb vector fields exist and highlighted the importance of Liouville-type vector fields, which emerge when Reeb fields are absent. Also, we were able to define dynamics on precontact manifolds. Furthermore, we showed that conformal rescaling can alter the class of the precontact structure, and we identified the precise conditions under which such changes occur.

Finally, we presented a definition of the evolution operator K within the framework of contact mechanics. This operator exhibits properties analogous to those of its classical counterpart and provides a fundamental link between the Lagrangian and Hamiltonian descriptions in the singular contact setting. To illustrate the theoretical developments, we examined explicit examples, including the conformal particle and the Cawley Lagrangian, both with added damping terms.

Future work There are several aspects of singular contact dynamics that deserve further investigation. For singular Lagrangian systems the operator K provides a tight connection between the Lagrangian and Hamiltonian formalisms. We plan to explore in detail these connections for contact Lagrangian systems. This includes also the development of constraint algorithms for contact Lagrangian systems and precontact Hamiltonian systems.

We believe that some ideas introduced in this thesis could be adapted to study singular Lagrangian systems in the (pre)cosymplectic and (pre)cocontact frameworks. This would make it possible to apply our methods to a broader class of time-dependent and dissipative systems.

A. Vector bundles

In this appendix, we present some theory of vector bundles and the associated geometric structures that arise naturally in this context. For further details, we refer the reader to [40, 53, 71].

A.1 Pull-back of a vector bundle

Let $\pi\colon E \to B$ be a vector bundle over a manifold B, and let $f\colon M \to B$ be a map from another manifold M to B. The *pull-back or induced bundle* $f^*(E)$ is defined as

$$f^*(E) := \{(x, e) \in M \times E \mid \pi(e) = f(x)\} .$$

It has a natural projection map

$$f^*(\pi)\colon f^*(E) \longrightarrow M\,, \qquad (x, e) \longmapsto x,$$

which gives $f^*(E)$ a vector bundle structure over M. The pull-back bundle is also usually called fibred product and denoted by $M \times_f E$. This construction also defines a (M, B)-vector bundle morphism

$$F\colon f^*(E) \longrightarrow E\,, \qquad (x, e) \longmapsto e\,.$$

The fibres of $f^*(E)$ are those of E, that is, for every $x \in M$, the map $F_x\colon f^*(E)_x \to E_{f(x)}$ is the identity.

The pull-back bundle $f^*(E)$ satisfies the following universal property: Given any vector bundle $\pi'\colon E' \to M$ and any (M, B)-vector bundle morphism $A\colon E' \to E$ there exists a *unique* vector bundle morphism $\widehat{A}\colon E' \to f^*(E)$ over M such that the following diagram commutes:

$$\begin{tikzcd}
E' \arrow[r, "\widehat{A}"] \arrow[d, "\pi'"'] \arrow[rr, bend left, "A"] & f^*(E) \arrow[r, "F"] \arrow[d, "f^*(\pi)"] & E \arrow[d, "\pi"] \\
M \arrow[r, equal] & M \arrow[r, "f"'] & B
\end{tikzcd}$$

That is, the vector bundle morphism A can be uniquely factorized as $A = F \circ \widehat{A}$.

There is a natural bijection between sections of $f^*(E)$ and *sections along* f. Indeed, every map $\sigma\colon M \to E$ such that $\pi \circ \sigma = f$, induces a section $\widetilde{\sigma} = (Id_M, \sigma)\colon M \to M \times_f E$. Similarly, if s is a section of E, then $s \circ f$ is a section along f.

A.2 The vertical bundle

Let $\pi\colon E \to Q$ be a vector bundle over a manifold Q. The *vertical bundle* is a canonical subbundle of the tangent bundle TE, consisting of the vectors in TE that are tangent to the fibres of the projection map π.

Namely, the vertical bundle, is defined as

$$V(E) := \mathrm{Ker}\,(\mathrm{T}\pi) \subset \mathrm{T}E\,.$$

That is, for each point $e \in E$, the vertical space at e is given by

$$V_e(E) := \operatorname{Ker}(\mathsf{T}_e\pi) \subset \mathsf{T}_e E\,,$$

Since π is a submersion, $\mathsf{T}_e\pi$ is surjective and its kernel defines a smooth distribution on E, which has the same rank as $\pi\colon E \to Q$. Its fibre at each point $e_x \in E_x$ is $V_{e_x}(E) = \mathsf{T}_{e_x}(E_x)$

In fact, for each $x \in Q$ and $e_x \in E_x$ we have that E_x is naturally isomorphic to $\mathsf{T}_{e_x}(E_x)$. One can construct such isomorphism by sending $v_x \in E_x$ to the tangent vector of the path $t \mapsto e_x + v_x$. This immediately implies that

$$\mathsf{T}(E_x) \cong E_x \times E_x\,,$$

which globally defines a canonical isomorphism $V(E) \cong E \times_\pi E$, called the *vertical lift*

$$\begin{aligned} \mathrm{vl}_E\colon E \times_\pi E &\longrightarrow V(E) \subset \mathsf{T}(E) \\ (e_x, v_x) &\longmapsto [t \mapsto e_x + t v_x]\,. \end{aligned}$$

The vertical lift defines a canonical bijection between vector bundle morphisms $E \to E$ and vertical vector fields on E. For any vector bundle morphism $\xi\colon E \to E$, we can define the vertical vector field

$$\xi^v\colon E \longrightarrow V(E) \subset \mathsf{T}E\,, \qquad \xi^v = \mathrm{vl}_E(e, \xi(e))\,.$$

In particular, if we apply this to the identity map $\mathrm{Id}_E\colon E \to E$, we obtain the so-called *Liouville vector field* $\Delta_E(e) = \mathrm{vl}_E(e,e)$. In local coordinates (q^i, a^i) for E, the Liouville vector field is expressed as $\Delta_E(x,a) = a^i \frac{\partial}{\partial a^i}$.

One can also construct the vector bundle morphism $\mu = (\tau_E, \mathsf{T}\pi)\colon \mathsf{T}E \to E \times_\pi \mathsf{T}Q$, to obtain the short exact sequence

$$0 \longrightarrow E \times_\pi E \xrightarrow{\mathrm{vl}_E} \mathsf{T}E \xrightarrow{\mu} E \times_\pi \mathsf{T}Q \longrightarrow 0\,,$$

which has $\operatorname{Im}(\mathrm{vl}_E) = \operatorname{Ker}(\mu) = V(E)$.

Let us now assume that $E = \mathsf{T}Q$, i.e. we consider the vector bundle $\tau_Q\colon \mathsf{T}Q \to Q$. In this case, the tangent bundle $\mathsf{TT}Q$ has to vector bundle structures over the same base $\mathsf{T}Q$, namely, $\mathsf{T}\tau_M\colon \mathsf{TT}Q \to \mathsf{T}Q$ and $\tau_{\mathsf{T}M}\colon \mathsf{TT}Q \to \mathsf{T}Q$. Both structures are canonically isomorphic through the so-called *canonical involution* $\kappa_Q\colon \mathsf{TT}Q \to \mathsf{TT}Q$, which has local expression

$$\kappa(x, v; u, a) = (x, u; v, a)\,.$$

In this particular case, one can compose the maps $J = \mathrm{vl}_{\mathsf{T}Q} \circ \mu$ to obtain the so-called *vertical endomorphism.* Locally, it is given by

$$J = \mathrm{d}q^i \otimes \frac{\partial}{\partial v^i} \qquad \text{or} \qquad J(q, v; u, a) = (q, v, 0, u)\,.$$

A.3 The map Γ

We still consider the case where $E = \mathsf{T}Q$, and let $\pi\colon P \to Q$ be a fibre bundle and $f\colon \mathsf{T}Q \to P$ a morphism of Q-fibre bundles, that is, the following diagram commutes

$$\begin{tikzcd} \mathsf{T}Q \arrow[r, "f"] \arrow[d, "\tau_Q"'] & P \arrow[dl, "\pi"] \\ Q & \end{tikzcd}$$

In this case, one can consider the following morphisms of $(\mathrm{T}Q)$-vector bundles

$$\mathrm{TT}Q \xrightarrow{\mathrm{T}\mathring{f}} \mathrm{T}Q \times_f \mathrm{T}P \xrightarrow{(\mathrm{Id},\mathrm{T}\pi)} \mathrm{T}Q \times_{\mathrm{T}Q} \mathrm{T}Q \xrightarrow{\mathrm{vl}_{\mathrm{T}Q}} \mathrm{TT}Q\,.$$

We can compose the second and third maps to get a morphisms of $(\mathrm{T}Q)$-vector bundles

$$\Gamma\colon \mathrm{T}Q \times_f \mathrm{T}P \to \mathrm{TT}Q\,,$$

with which, given any vector field Y along f, we can construct a vertical vector field on $\mathrm{T}Q$, as

$$\Gamma \circ Y = \mathrm{vl}_Q \circ (\mathrm{Id}_{\mathrm{T}Q}, \mathrm{T}\pi \circ Y)\,.$$

Assume that, locally, $f(q,v) = (q, \widehat{p}(q,v))$, then we have

$$\Gamma(q,v;q,\widehat{p},u,k) = (q,v;0,u)\,.$$

Lastly, let us consider two particular cases.

Assume that we have a *symplectic* manifold (P,ω). Given a function $h \in \mathscr{C}^\infty(P)$, there exists a unique hamiltonian vector field $X_h \in \mathfrak{X}(P)$. The map $X_h \circ f$ is a vector field along f, so we can apply Γ to it to obtain a vector field on $\mathrm{T}Q$

$$\Gamma_h = \Gamma \circ X_h \circ f\,.$$

In canonical coordinates, its local expression is

$$\Gamma_h = f^*\left(\frac{\partial h}{\partial p}\right)\frac{\partial}{\partial v}\,.$$

We can do an analogous construction if we have a *contact* manifold (P,η). In this case, given a function $h \in \mathscr{C}^\infty(P)$ we can also define a (unique) contact Hamiltonian vector field

$$X_h = B^{-1}(\mathrm{d}h - (\mathscr{L}_R h)\eta)\,.$$

Again, from this we can define $X_h \circ f$, a vector field along f, which composed with Γ gives the vector field on $\mathrm{T}Q$

$$\Gamma_h = \Gamma \circ B^{-1}(\mathrm{d}h - (\mathscr{L}_R h + h)\eta) \circ f\,.$$

In canonical coordinates, the vector field has local expression

$$\Gamma_h = f^*\left(\frac{\partial h}{\partial p}\right)\frac{\partial}{\partial v}\,.$$

If one instead considers the vector field

$$B^{-1}(\mathrm{d}h) = X_h + (\mathscr{L}_R h + h)R\,,$$

the local expression of the vector field $\Gamma \circ B^{-1} \circ f$ on $\mathrm{T}Q$ is still the same

$$\Gamma \circ B^{-1}(\mathrm{d}h) \circ f = f^*\left(\frac{\partial h}{\partial p}\right)\frac{\partial}{\partial v}\,.$$

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